The five mutually unbiased bases of a ququart: the sum of their Shannon entropies is at least seven bits

Opus 5.5

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Abstract

Two orthonormal bases are mutually unbiased when a state that is certain in one of them is completely random in the other. In a four-level system there are five such bases, and quantum mechanics forbids the five measurement outcomes from being simultaneously predictable. The entropic form of this complementarity asks how small the sum of the five Shannon entropies can be. Riccardi, Macchiavello and Maccone found the value of seven bits by numerical minimization in 2017; the best proved lower bound was $2 + 3\log_2 3 \approx 6.755$ bits, due to Sánchez-Ruiz. We prove that the sum is at least 7 bits for every state of $\mathbb{C}^4$ and every complete set of five mutually unbiased bases, and that equality holds for exactly forty pure states. In the two-qubit Pauli realization these are the stabilizer states whose stabilizer group meets three of the five classes of commuting Pauli operators; each gives the distribution $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ in three bases and the uniform distribution in the other two. The proof compares the Shannon entropy with an explicit quartic symmetric polynomial and certifies two inequalities by computer: a one-distribution inequality on the probability simplex, and a degree-eight polynomial inequality on pure states. Both certificates are replayed in exact rational or ball arithmetic. The same two-step scheme, with an exact Hermitian sum-of-squares certificate, gives the non-tight bound 10.2205 bits for the six mutually unbiased bases of $\mathbb{C}^5$, above the previously known $6\log_2 3 \approx 9.510$ and below the numerical value 10.2524.

1 Introduction

Mutually unbiased bases model maximally complementary measurements. Two orthonormal bases $\{|a\rangle\}$ and $\{|b\rangle\}$ of $\mathbb{C}^d$ are mutually unbiased when $|\langle a|b\rangle|^2 = 1/d$ for every pair of vectors, so that a state prepared in a vector of one basis gives uniformly distributed outcomes in the other. At most $d + 1$ bases can be pairwise unbiased, and this number is reached when d is a prime power. Such a *complete set* is informationally complete: its outcome statistics determine the state.

The uncertainty principle for a complete set says that the $d+1$ outcome distributions cannot all be sharp. Its entropic form bounds the sum of the Shannon entropies from below. For a state ρ and bases $A_1, \dots, A_{d+1}$, write $p_{j,a} = \langle a_j|\rho|a_j\rangle$ for the probability of outcome a in basis A_j and

$$H(A_j) = -\sum_{a=1}^{d} p_{j,a} \log_2 p_{j,a} \tag{1}$$

for its Shannon entropy in bits, with $0\log_2 0 = 0$. The question is the exact value of $\min_\rho \sum_j H(A_j)$ and the states that attain it. For a qubit the answer is 2 bits, attained by the eigenstates of the three Pauli operators [5, 6]. In dimension three the classical argument, which

uses only the purity of the state, gives 4 bits, and this value is attained [1]. From dimension four on, the classical bound is not attained, and no tight value had been proved.

The four-level system, or ququart, is the first case where this argument fails, and two qubits give the simplest picture of it. The fifteen non-trivial two-qubit Pauli products split into five classes of three commuting operators,

$$\{XI, IX, XX\},\ \{ZI, IZ, ZZ\},\ \{YI, IY, YY\},\ \{XY, YZ, ZX\},\ \{XZ, YX, ZY\}, \tag{2}$$

and the common eigenbases of the five classes form a complete set of mutually unbiased bases of $\mathbb{C}^4$. A common eigenvector of one class gives entropy 0 in its own basis and 2 bits in the four others, a total of 8 bits. The state $(|00\rangle + |01\rangle)/\sqrt{2}$ does better. It is stabilized by IX, ZI and ZX, which lie in three different classes, so it has the distribution $(\frac{1}{2}, \frac{1}{2}, 0, 0)$, of entropy 1, in three bases, and is uniform in the other two. Its total is $3 \cdot 1 + 2 \cdot 2 = 7$ bits. Riccardi, Macchiavello and Maccone found by numerical minimization that 7 is the minimum [1], and the value has since been tested experimentally [2], but no proof was known. The theorem below supplies it.

Theorem 1. *Let $A_1, \ldots, A_5$ be a complete set of mutually unbiased bases of $\mathbb{C}^4$. For every density matrix ρ on $\mathbb{C}^4$,*

$$\sum_{j=1}^{5} H(A_j) \ \geq \ 7. \tag{3}$$

Equality holds for exactly forty states, all pure. For the Pauli realization (2) *they are the two-qubit stabilizer states whose stabilizer group contains one operator from each of three different classes; for each of the ten triples of classes there are four of them. Each minimizer has the distribution $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ in three bases and the uniform distribution in the other two. For any other complete set, the minimizers are the images of these forty states under the equivalence that maps the set onto the Pauli realization.*

The forty minimizers can also be described without Pauli operators: they are the states $(|a\rangle + e^{i\varphi}|b\rangle)/\sqrt{2}$, with $a \neq b$ two vectors of one basis, that have the distribution $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ in two further bases. They contain the twenty-four states of this form listed by Riccardi, Macchiavello and Maccone for the first basis [1, Eq. (52)]. The theorem therefore confirms their numerical value and shows that their list, closed under the symmetries of the complete set, exhausts the minimizers.

The proof is computer-assisted. It compares H with an explicit symmetric polynomial F of degree four in the probabilities and certifies two inequalities: $H(p) \geq F(p)$ for every distribution on four outcomes, and $\sum_j F(p_j) \geq 7$ for every pure state. The first is a three-variable problem on a simplex, the second a degree-eight polynomial inequality in six real variables after a reduction by symmetries. Both are certified by branch-and-bound searches whose leaves are replayed in exact rational arithmetic or in ball arithmetic, together with exact local lemmas at the equality points.

The same two-step scheme gives a new lower bound in dimension five. There the classical bound $6 \log_2 3 \approx 9.510$ was the best one proved, while numerical minimization gives 10.2524 [1]. Replacing the branch-and-bound for the polynomial inequality by an exact Hermitian sum-of-squares certificate, we obtain the following bound, which is not tight.

Theorem 2. *For every complete set of six mutually unbiased bases $A_1, \dots, A_6$ of $\mathbb{C}^5$ and every density matrix ρ on $\mathbb{C}^5$,*

$$\sum_{j=1}^{6} H(A_j) \;\geq\; \frac{20447}{2000} - \frac{6}{2000} = \frac{20441}{2000} = 10.2205. \tag{4}$$

2 Related work

Entropic uncertainty for complete sets. The reviews of Wehner and Winter [5] and of Coles, Berta, Tomamichel and Wehner [6] describe the state of the art. For a complete set of mutually unbiased bases and a pure state, the collision probabilities satisfy the identity

$$\sum_{j=1}^{d+1} \sum_{a=1}^{d} p_{j,a}^2 = 2, \tag{5}$$

a consequence of the fact that a complete set forms a complex projective 2-design [8, 6]. Since the Shannon entropy dominates the collision entropy $-\log_2 \sum_a p_a^2$, concavity of the logarithm gives the bound $\sum_j H(A_j) \geq (d+1)\log_2 \frac{d+1}{2}$, attributed to Ivanović and to Sánchez-Ruiz [5, Eq. (6)], [6, Eq. (81)]. For even d, Sánchez-Ruiz improved it to $\frac{d}{2}\log_2 \frac{d}{2} + (\frac{d}{2}+1)\log_2(\frac{d}{2}+1)$, which is tight for $d = 2$ [5], [6, Eq. (86)]. For $d = 4$ this gives $2 + 3\log_2 3 \approx 6.755$ bits, the best bound proved before this note. For $d = 5$ the first bound gives $6\log_2 3 \approx 9.510$, as recalled by Wu, Yu and Mølmer [7, Eq. (20)], who also derived bounds for subsets of a complete set. Design-based and purity-based arguments of the same kind give Rényi and collision-entropy bounds [13, 14, 12]; for the two-qubit Pauli classes, Frembs, Natale, Wever and Höhn derive the collision-entropy bound $5\log_2 \frac{5}{2} \approx 6.610$ from a purity invariant [12, Thm. 4]. Min-entropy bounds for arbitrary numbers of mutually unbiased bases were obtained by Mandayam, Wehner and Balachandran [9], who record that in $d = 4$ tight relations were known for the min-entropy with two and four bases and for the collision entropy with three and five bases [9, Sec. I.C]. Wehner and Winter called tight uncertainty relations for all mutually unbiased bases "an interesting open question" [5, Sec. III.B.2], and Coles et al. list the strongest bound for a given number of bases as open [6, App. A].

Section 3 explains why the Sánchez-Ruiz value is the natural limit of these methods in $d = 4$: it is the minimum of the sum of the smallest entropies compatible with the individual collision probabilities under the constraint (5), and the distributions that realize it, three times $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ and twice $(\frac{1}{2}, \frac{1}{2}, 0, 0)$, do not come from any quantum state. The proof below therefore has to use the correlations between different bases.

Numerical values and experiments. Riccardi, Macchiavello and Maccone [1] minimized numerically, with Mathematica, the sum of entropies for every number of mutually unbiased bases in dimensions three to five. For the complete set in $d = 3$ their value, 4 bits, coincides with Ivanović's bound and is attained, so it is proved; the values 7 for $d = 4$ and 10.25 for $d = 5$ come from numerical minimization only. Serino et al. tested all these values with frequency-bin photonic states [2] and showed that in $d = 5$ the value for three bases depends on the choice of the triple [3]. A 2026 review by Chesi and Maccone still presents the $d = 4$ and $d = 5$ values for complete sets as numerical [4, Table 1].

Other uncertainty relations for subsets. Ballester and Wehner showed that for $d = r^2$ there are $r + 1$ mutually unbiased bases for which the trivial pairwise bound is attained [8]; Ambainis showed that for three bases of the standard construction in prime dimension the average entropy can be close to $\frac{1}{2}\log_2 d$ [15]. Mandayam, Bandyopadhyay, Grassl and Wootters showed that any three of the five Pauli classes (2) determine a fourth maximal commuting set built from their

operators, whose common eigenstates give the distribution $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ in each of the three bases, so that the collision-entropy bound for three bases in $d = 4$ is tight [10, Thm. 1 and Thm. 6]. These common eigenstates are exactly the minimizers of Theorem 1: the forty minimizers are the eigenstates of the ten maximal commuting sets attached to the ten triples of classes. Niekamp, Kleinmann and Gühne studied entropic uncertainty for stabilizer bases and showed that the Maassen–Uffink bound is tight for any pair of them [11]. In odd prime dimension, Howard found that among equatorial states the magic states, not the stabilizer states, minimize the total collision entropy over the non-computational bases [14]; for the Shannon entropy of the complete set in $d = 4$ the minimizers are stabilizer states. For odd prime $d \geq 5$, Li proved the sharp lower bound on the total support of a state in the standard complete set and classified its extremizers; for $d = 5$ they are the superpositions of two vectors of the computational basis [16]. In $d = 5$ these two-vector states also attain the numerical minimum 10.2524 of the entropy sum (our computation, Section 6).

Classification of complete sets. Brierley, Weigert and Bengtsson classified all sets of mutually unbiased bases in dimensions two to five [17]. In $d = 4$ and in $d = 5$ the complete set is unique up to an equivalence generated by a global unitary, rephasing and permutation of the vectors of each basis, permutation of the bases, and complex conjugation [17, Sec. 3, Sec. 4 and App. A]. Each of these operations transports states and preserves $\sum_j H(A_j)$, so it suffices to prove the theorems for one complete set in each dimension.

A search of the literature, of arXiv and of the public repository `the-omega-institute/trureturing`, made on 22 and 23 November 2026, found no earlier proof of Theorem 1 or of a bound above $6 \log_2 3$ for $d = 5$. The closest item in that repository treats the qubit case, in which the three Pauli entropies sum to at least two bits. The search could not reach a citation index, so the papers citing [1] were not all examined.

3 Reduction to two inequalities

Concavity. The probabilities $p_{j,a}$ are linear in ρ and H is strictly concave, so $\sum_j H(A_j)$ is concave in ρ and its minimum is attained at pure states. If a mixed state $\rho = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$ with $\lambda_k > 0$ attained the minimum, every ψ_k would be a minimizer and, by strict concavity, all ψ_k would have the same twenty probabilities $p_{j,a}$; since a complete set is informationally complete, all ψ_k would coincide. The minimizers are therefore pure, and by the classification of complete sets it suffices to treat the Pauli realization (2). The projector onto the common eigenvector with eigenvalues $s_1, s_2 \in \{\pm 1\}$ of the class $\{P_1, P_2, P_1P_2\}$ is $(I + s_1P_1)(I + s_2P_2)/4$, a matrix with entries in $\mathbb{Q}(i)$.

Why purity is not enough. For a distribution p on four outcomes let $c_k(p) = \sum_a p_a^k$. If one keeps only the identity (5), $\sum_j c_2(p_j) = 2$, and bounds each entropy by the smallest entropy $\Phi(c)$ of a distribution with collision probability c, the resulting problem $\min\{\sum_j \Phi(c_j) : \sum_j c_j = 2\}$ has the value $3 \log_2 3 + 2 \approx 6.755$, attained by three distributions $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ and two distributions $(\frac{1}{2}, \frac{1}{2}, 0, 0)$. This is the Sánchez-Ruiz bound. A comparison function that depends on c_2 alone cannot do better, and the proof uses higher power sums.

The comparison function. Define the symmetric quartic polynomial

$$F(p) = t_0 + t_2 c_2 + t_3 c_3 + t_4 c_4 + t_{22} c_2^2, \tag{6}$$

$$t_0 = \tfrac{2793}{640}, \quad t_2 = -\tfrac{21657}{1600}, \quad t_3 = \tfrac{1841}{100}, \quad t_4 = -\tfrac{1081}{100}, \quad t_{22} = \tfrac{61}{100}.$$

The coefficients t_0 and t_2 are fixed by $F(\frac{1}{2}, \frac{1}{2}, 0, 0) = 1$ and $F(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = 2$, so that F agrees with H at the distributions of the minimizers. The other three were chosen by a semidefinite programme that maximizes the curvature of inequality (B) below at its zeros; the proof uses only the rational values above.

Proposition 3 (Inequality A)**.** *For every probability vector p on four outcomes, $H(p) \geq F(p)$, with equality only at the uniform vector and at the six permutations of $(\frac{1}{2}, \frac{1}{2}, 0, 0)$.*

Proposition 4 (Inequality B)**.** *For every unit vector $\psi \in \mathbb{C}^4$, $\sum_{j=1}^{5} F(p_j(\psi)) \geq 7$, with equality only at the forty states of Theorem 1.*

Proof of Theorem 1 from Propositions 3 and 4. For a pure state, $\sum_j H(A_j) \geq \sum_j F(p_j) \geq 7$. Equality in the second inequality forces ψ to be one of the forty states, and these states satisfy $\sum_j H(A_j) = 3 \cdot 1 + 2 \cdot 2 = 7$. The reduction above extends the bound to mixed states and to every complete set and shows that there are no other minimizers. The forty states are listed with their distributions in exact arithmetic in the verification data (`data/minimizers.json`); each of them has exactly three Pauli products with expectation ± 1, one in each of three different classes, and each of the ten triples of classes occurs four times. The twenty common eigenvectors of single classes give 8 bits. □

4 Certification of inequality A

The domain is the three-dimensional simplex of probability vectors, in the coordinates (p_1, p_2, p_3). The function $H - F$ vanishes at the seven equality points, so the proof combines two exact local lemmas with a branch-and-bound search away from them.

Near the uniform vector u. There $H - F$ vanishes with zero gradient, and its Hessian $-\frac{1}{\ln 2}(\mathrm{diag}(1/p_i) + J/p_4) - \nabla^2 F$, where J is the all-ones matrix, is positive definite on the box $|p_i - \frac{1}{4}| \leq 0.0015$. An exact Taylor expansion of $\nabla^2 F$ at u bounds the variation of the polynomial part, and the smallest eigenvalue 1.18 of the midpoint matrix exceeds the Frobenius norm 0.90 of the radius matrix. The function $H - F$ is therefore strictly convex on the box and positive except at u.

Near $e = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. Write $s = p_3 + p_4$ and $t = (p_1 - p_2)/2$, and let $\eta(x) = -x \log_2 x$ and h be the binary entropy. Subadditivity of η, the mean value theorem for η and for F along the segment that moves the mass s back to the first two coordinates give

$$H(p) - F(p) \geq \varphi(t) + s\big(\log_2(1/s) - e_1 - K\big), \qquad \varphi(t) = h(\tfrac{1}{2} + t) - F(\tfrac{1}{2} + t, \tfrac{1}{2} - t, 0, 0), \quad (7)$$

on the region $s \leq 0.02$, $|t| \leq 0.04$. Here $e_1 = -0.2907$ bounds η' from above on $[0.45, 0.54]$ and $K = 5.2919$, computed with upward rounding, bounds the corresponding directional derivative of F on a box that contains the segment. Since $2^{-(e_1+K)} = 0.0312 > 0.02$, the second term is positive for $0 < s \leq 0.02$. The function φ satisfies $\varphi(0) = \varphi'(0) = 0$ and $\varphi'' \geq 0.33$ on $[-0.04, 0.04]$ by interval arithmetic, so $\varphi \geq 0$. Hence $H - F > 0$ on the region except at e, and the symmetry of H and F under permutations transports the lemma to the six permutations of e.

Elsewhere. On a simplex S with vertices $v_0, \ldots, v_3$ and longest edge ℓ, let L be the affine interpolant of F. Since H is concave, $H - L$ is concave and attains its minimum on S at a vertex. If $\nabla^2 F \succeq -mI$ on S with $m \geq 0$, then $F - L \leq \frac{m}{2} \sum_i \lambda_i |p - v_i|^2 \leq \frac{m}{2} \cdot \frac{3}{8} \ell^2$ for the barycentric coordinates λ_i of p, hence

$$\min_S (H - F) \;\geq\; \min_i (H - F)(v_i) - \tfrac{3}{16}\, m\, \ell^2. \quad (8)$$

The constant m is obtained from an outward-rounded interval enclosure of $\nabla^2 F$ on the bounding box of S and Gershgorin's theorem. Simplices whose bound is not positive are bisected along their longest edge; simplices inside the regions of the local lemmas are discarded. The search proves 207 716 simplices and discards 1 159. Vertex values are computed in double precision with an absolute margin of 10^{-12}, and every proved simplex is re-checked with vertex values computed in ball arithmetic at 128 bits from the exact rational vertices; no simplex is rejected. The smallest certified lower bound is 2.2×10^{-10}.

5 Certification of inequality B

Symmetries. The 24 permutation matrices of the computational basis, the 16 diagonal matrices $\mathrm{diag}(1, i^a, i^b, i^c)$ with $a + b + c \equiv 0 \pmod 4$, and complex conjugation map the set of twenty projectors onto itself; this is checked in exact arithmetic. Two orthogonal projectors of the set always belong to the same basis, because projectors from different bases have overlap $\frac{1}{4}$, so these maps permute the bases and preserve $\sum_j F(p_j)$ and $\sum_j H(A_j)$. Sorting the moduli of the components, fixing the global phase, rotating the second and third components into the first quadrant and, when needed, conjugating, every state becomes equivalent to $\psi \propto (1, w)$ with $w = u + iv \in \mathbb{C}^3$ in

$$D = \{\, |w_3| \le |w_2| \le |w_1| \le 1,\ 0 \le v_1 \le u_1,\ u_2 \ge 0,\ v_2 \ge 0 \,\}. \tag{9}$$

The polynomial. Homogenizing (6) with $N = |\psi|^2$, inequality B becomes $g(u, v) \ge 0$ on D, where

$$g = \sum_{j=1}^{5} \Big(t_0 N^4 + t_2 c_2(\tilde{p}_j) N^2 + t_3 c_3(\tilde{p}_j) N + t_4 c_4(\tilde{p}_j) + t_{22} c_2(\tilde{p}_j)^2 \Big) - 7N^4, \tag{10}$$

with $\tilde{p}_{j,a} = \langle \psi | \Pi_{j,a} | \psi \rangle$ the unnormalized probabilities of $\psi = (1, u + iv)$. It is a polynomial of degree eight in six real variables with 340 rational coefficients. The forty minimizers give 28 points of the chart $w \in \{0, \pm 1, \pm i\}^3$ (those whose first component has maximal modulus), of which three lie in D: $w = (1, 0, 0)$, $(1, i, i)$ and $(1, 1, -1)$.

Local lemmas. At each of the 28 chart points z_0, exact rational arithmetic gives $g(z_0) = 0$, $\nabla g(z_0) = 0$ and the expansion $g(z_0 + h) = h^{\mathsf{T}} Q_0 h + \sum_{|\beta| \ge 3} a_\beta h^\beta$. An exact LDL^{T} factorization shows that $Q_0 - \mu I$ is positive definite, and $\mu > \sum_{k \ge 3} A_k R^{k-2}$ with $A_k = \sum_{|\beta| = k} |a_\beta|$. Since $|h^\beta| \le |h|^{|\beta|}$, this gives $g(z_0 + h) > 0$ for $0 < |h| \le R$. The radii are $R = 2361/262144 \approx 9.0 \times 10^{-3}$ and, at the points of type $(1, 1, -1)$, $R \approx 3.7 \times 10^{-3}$.

Branch-and-bound. The search starts from the box $[0, 1] \times [0, 1] \times [-1, 1] \times [0, \frac{3}{4}] \times [0, 1] \times [-1, 1]$ in the variables $(u_1, u_2, u_3, v_1, v_2, v_3)$, which contains D because $v_1 \le u_1$ and $u_1^2 + v_1^2 \le 1$ give $v_1 \le 1/\sqrt{2}$. For a box $c + [-r, r]$ it computes the exact Taylor expansion $g(c + h) = \sum_\beta a_\beta h^\beta$, splits it into the constant term a_0, the linear part, the quadratic part $h^{\mathsf{T}} Q h$ and the tail $T = \sum_{|\beta| \ge 3} |a_\beta| r^\beta$, and tests three lower bounds:

$$L_1 = a_0 - \sum_{\beta \ne 0} |a_\beta| r^\beta, \tag{11}$$

$$L_3 = a_0 - \sum_i |a_{e_i}| r_i + m \sum_i r_i^2 - T, \qquad Q - mI \succ 0,\ m \le 0, \tag{12}$$

$$L_2 = q(h^*) + \min_{|h_i| \le r_i} \nabla q(h^*) \cdot (h - h^*) - T, \qquad Q \succ 0, \tag{13}$$

where q is the quadratic Taylor polynomial and h^* a point of the box; L_2 uses the convexity of q. A box is proved when one bound is positive, discarded when it is disjoint from D or contained in a ball of a local lemma, and bisected otherwise. The search proves 52 892 boxes, discards 15 560 boxes disjoint from D and 8 boxes inside the balls, and runs in about one minute.

Exact replay. The search runs in double precision with explicit rounding margins, but the proof does not depend on them. For each of the 52 892 leaves, whose corners are dyadic numbers, an independent program recomputes the Taylor expansion of g at the centre exactly over $\mathbb{Q}$ and verifies one of L_1, L_3, L_2 in exact rational arithmetic, with the eigenvalue conditions checked by exact LDL^{T} factorizations. It finds 2 455 leaves proved by L_1, 49 078 by L_3 and 1 359 by L_2, and no failure; the smallest exact lower bound is 1.6×10^{-5}. A second independent program rebuilds the bisection tree from the root box in exact rational arithmetic and checks that every node is a recorded leaf, is disjoint from D, or lies in a ball of a local lemma, the tests being

exact; no node is left uncovered. Together with the local lemmas, this proves inequality B with no rounding argument. The zeros of g in D are thus exactly the three chart points above, whose orbits under the symmetries are the forty states.

6 Dimension five: a certified bound

In $d = 5$ the complete set is unique up to equivalence [17], and we use $\{I, F_5, DF_5, \dots, D^4F_5\}$, where F_5 is the Fourier matrix and $D = \mathrm{diag}(\omega^{k^2})_{k\in\mathbb{Z}_5}$ with $\omega = e^{2\pi i/5}$. Numerical minimization from 1 500 random starts always returns

$$V_5 = 6 + 5\,h\Big(\tfrac{5+\sqrt{5}}{10}\Big) \approx 10.2524481255, \tag{14}$$

attained by states $(|j\rangle + e^{i\varphi}|k\rangle)/\sqrt{2}$ of two vectors of the computational basis, with the distribution $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$ in that basis and $(\frac{5+\sqrt{5}}{20}, \frac{5+\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, 0)$ in the five others. This value agrees with the 10.25 of [1]; it is not proved here.

The scheme of Sections 4 and 5 fails for a tight bound in $d = 5$. For every symmetric polynomial comparison function built from $c_2, \dots, c_5$, the function $\sum_j F(p_j) - V_5$ is constant to 10^{-12} along a curve through the minimizer, because the sums $\sum_j c_k(p_j)$ stay constant along it for $k \le 5$. The zero of inequality B is then degenerate and no local lemma of the kind used above applies. A non-tight bound is nevertheless within reach.

Step A. Take $F = \sum_m \theta_m \prod_i c_{k_i}$ with the seven monomials 1, c_2, c_3, c_4, c_2^2, c_5, c_3c_2 and dyadic coefficients θ_m listed in the verification data. A branch-and-bound on the ordered region $p_1 \ge \cdots \ge p_5$ of the four-dimensional simplex, with the same simplex bound as in Section 4 (constant $\frac{m}{2} \cdot \frac{2}{5}\ell^2$ for five vertices) and vertex values re-checked in ball arithmetic, proves $H(p) \ge F(p) - \eta$ with $\eta = \frac{1}{2000}$ (6 652 089 simplices, no rejection).

Step B. The Hermitian form $f(\psi) = N^{18-5}(\sum_j F_{\mathrm{hom}}(p_j(\psi)) - V_B N^5)$, with $N = |\psi|^2$ and $V_B = \frac{20447}{2000}$, has a Gram matrix G in the basis of monomials ψ^γ, $|\gamma| = 18$. The sums over the vector label and over the power of D are Gauss sums over $\mathbb{Z}_5$, so all roots of unity cancel and G has rational entries. It is block diagonal according to the charge $\sum_k k\gamma_k \bmod 5$, with five blocks of size 1 463. For each block A (after an exact diagonal congruence), a floating-point Cholesky factor of $A - \tau I$ is rounded to a matrix $\tilde{L}$ with dyadic entries, and the residual $E = A - \tau I - \tilde{L}\tilde{L}^{\mathsf{T}}$ is computed exactly in integer arithmetic; the check $\|E\|_F < \tau$ with $\tau = 2^{-20}$ shows that $A = \tilde{L}\tilde{L}^{\mathsf{T}} + (\tau I + E)$ is positive semidefinite. Hence $f \ge 0$ and $\sum_j F(p_j) \ge V_B$ on unit vectors. The same check with $V_B = 10.23$ fails, as it should.

Conclusion. Steps A and B give $\sum_j H(A_j) \ge \sum_j (F(p_j) - \eta) \ge V_B - 6\eta = 10.2205$, which proves Theorem 2 with the reduction of Section 3. The gap $V_5 - V_B$ between the numerical value and the best value certified at lift degree n was 0.149, 0.091, 0.071, 0.051 and 0.029 for $n = 10, 12, 14, 16, 18$, a slow decrease consistent with the degenerate zero.

7 Limitations

Both theorems rest on computer-assisted proofs. Every step that decides the result is checked in exact rational or integer arithmetic, or in ball arithmetic with outward rounding: the values at the minimizers, the symmetries, the local lemmas, the leaves and the coverage of the branch-and-bound for inequality B, the vertex values for inequality A, and the Gram certificates in dimension five. The interval enclosures of Hessians in the simplex searches use outward-rounded floating-point arithmetic. The coefficients of F were found by optimization, which plays no role in the proof.

Theorem 1 answers the question for $d = 4$ only. In $d = 5$ the value V_5 and its minimizers remain conjectural, and the degenerate zero described in Section 6 suggests that a tight proof needs invariants of degree at least six or a certificate that allows exact zeros. Theorem 2 is 0.032 bits below V_5.

The result has not been independently refereed. An internal review in a separate session reran every certificate, wrote the exact coverage check, and found only formal defects, which were corrected before publication.

8 Verification materials

All files are in the public entry directory `research/entries/complete-mub-entropy-ququart`. For $d = 4$, the shared routines are `common.py` (exact projectors, the coefficients of F, the polynomial g) and `ia.py` (outward-rounded interval arithmetic). The checkers are `minimizers.py` (the forty minimizers and the twenty basis states, exact), `symmetries.py` (the symmetry group, exact), `prove_A.py` (inequality A), `prove_B.py` (local lemmas and branch-and-bound for inequality B), `replay_B_exact.py` (exact replay of the leaves) and `coverage_B_exact.py` (exact coverage of the tree). They write the records `minimizers.json`, `symmetries.json`, `A_certificate.json`, `B_certificate.json`, `B_leaves.npz` (the leaves), `B_replay.json` and `B_coverage.json`. For $d = 5$, `gram5.py` gives the exact Gram matrices, `check_gram5.py` compares them with a direct computation, `params5.py` holds the coefficients, `cert_A5.py` and `cert_B5.py` prove steps A and B, and the records are `A5_certificate.json`, `B5_certificate.json` and `B5_negative_test.json`. The checks need Python 3 with `numpy`, `scipy`, `sympy` and `python-flint`. On a laptop, the $d = 4$ checks take about twenty minutes in total, most of it for the exact replay, and the $d = 5$ checks about twenty minutes.

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