Exact PPT entanglement distillation from the Zhu–Wang qutrit support: the one-copy optimum and the first four output dimensions

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Abstract

Entanglement distillation converts many copies of a noisy entangled state into fewer, perfectly entangled ones. When the conversion must be exact and the allowed operations are the positive-partial-transpose (PPT) operations, whose partial transpose is again completely positive, the best output from a fixed number of copies is the solution of a semidefinite program that depends only on the support of the state. Zhu and Wang recently exhibited a rank-three support P on two qutrits for which the asymptotic rate of exact PPT distillation lies strictly below the min-Rains relative entropy, its standard upper bound; they bound the rate from above by a witness but do not evaluate the one-copy program. We solve that program in closed form. The optimal value is $W_0(P) = \sqrt{177}/(7+\sqrt{177}) = (177-7\sqrt{177})/128 \approx 0.65524$, so a single copy cannot yield even one exact Bell pair. The proof reduces the problem by a phase symmetry of P to six real parameters and solves the reduced problem exactly. Combined with the witness of Zhu and Wang, the result fixes the largest exact maximally entangled output obtainable from n copies of any state supported on P: its dimension is 1, 2, 3 and 5 for $n = 1, 2, 3, 4$. Four copies therefore give a five-dimensional maximally entangled state, more than the dimension four obtained from two separate two-copy runs. The asymptotic rate lies in $[\log_2(1 + 7/\sqrt{177}), \log_2(391/250)) \approx [0.6099, 0.6452)$.

1 Introduction

Distillation is the basic test of whether the entanglement in a mixed state is usable: Alice and Bob receive n copies of a bipartite state ρ and try to produce a maximally entangled state $\Phi_M = \frac{1}{M}\sum_{i,j=0}^{M-1} |ii\rangle\langle jj|$ of the largest possible Schmidt rank M [3]. In the exact (zero-error) version of the task, the output must equal Φ_M with certainty. Operations of local quantum processing with classical communication are hard to characterize, and a standard relaxation allows every operation whose partial transpose is also completely positive (PPT operations) [2, 3]. PPT states cannot be distilled at all [1], so the relaxation keeps the essential obstruction while turning the optimization into a semidefinite program.

For exact PPT distillation from a state with support projection P, the program reads

$$W_0(P) = \min\{\|E^\Gamma\|_\infty : \ P \le E \le \mathbf{1}\}, \tag{1}$$

where E^Γ is the partial transpose and $\|\cdot\|_\infty$ the operator norm; Φ_M is reachable from one copy exactly when $M\,W_0(P) \le 1$, so the best integer output is $\lfloor W_0(P)^{-1} \rfloor$ [4, 8]. The asymptotic rate $E^\infty_{0,\mathrm{PPT}}(\rho)$ is the limit of $-\frac{1}{n}\log_2 W_0(P^{\otimes n})$. It is bounded above by the min-Rains relative

entropy $R_{\min}$ [5]. Whether this bound is tight was asked by Wang and Duan [5]; Zhu and Wang have shown that it is not [8]. Their argument uses the fact that every feasible E acts as the identity on the support, and a range-supported witness T that gives $W_0(P^{\otimes n}) \geq \|T^\Gamma\|_1^{-n}$ for every n. For their explicit qutrit support the witness bound lies strictly below $R_{\min}$.

The witness controls the problem from one side only. This note answers the complementary question for the same support: what is $W_0(P)$ itself, and what can actually be distilled from a few copies? The answer is exact. The support P is invariant under the phase rotations $U \otimes U$ with U diagonal; averaging over them makes the optimal effect block diagonal, the remaining six parameters decouple into a scalar inequality, and

$$W_0(P) = \lambda_* := \frac{\sqrt{177}}{7+\sqrt{177}} = \frac{177-7\sqrt{177}}{128} \approx 0.6552426.$$

The value is attained by an explicit effect E_* with $\|E_*^\Gamma\|_\infty = \lambda_*$. Because $E_*^{\otimes n}$ is feasible for n copies, the maximal output dimension M_n satisfies $M_n \geq \lfloor \lambda_*^{-n} \rfloor$, and by the witness of Zhu and Wang $M_n < (391/250)^n$. These two bounds have the same integer part for $n \leq 4$, which gives $M_1, \ldots, M_4 = 1, 2, 3, 5$ exactly for every state with support P.

2 Related work

The Horodecki family showed that states with positive partial transpose are undistillable, which makes the partial transpose the natural tool for upper bounds [1]. Rains extended the relative-entropy bound on distillable entanglement to PPT operations [2] and then cast the fidelity of PPT distillation as a semidefinite program, which yields the semidefinite upper bound now called the Rains bound [3].

Wang and Duan gave the semidefinite programs for exact PPT distillation that underlie the formulation (1) [4]. In a second paper they introduced the bound later called the min-Rains relative entropy, showed numerically that the Rains bound is not additive, and asked in their conclusion whether the min-Rains relative entropy equals the exact PPT distillable entanglement [5]. For pure states the exact question has a complete answer: Matthews and Winter showed that the exact PPT distillable entanglement of a pure state is the min-entropy of its Schmidt coefficients [7, Prop. 4]. Fang, Wang, Tomamichel and Duan developed the one-shot and second-order theory of PPT distillation at nonzero error; at zero error their one-shot quantity reduces to the program of Wang and Duan [6, 4]. Our reformulation of the feasibility condition $MW_0(P) \leq 1$ and of the integer output $\lfloor W_0(P)^{-1} \rfloor$ follows the presentation of Zhu and Wang, who attribute it to [4].

The present note depends directly on Zhu and Wang [8]. They proved that every feasible effect satisfies $EP = PE = P$, constructed the range-supported witness and the bound $W_0(P^{\otimes n}) \geq \|T^\Gamma\|_1^{-n}$ (their Theorem 2.2), and gave the explicit qutrit support P used here together with the rational certificate $\|T^\Gamma\|_1 < 391/250$ and the lower bound $R_{\min} \geq -\log_2(6393/10000) \approx 0.6454$ (their equation (3.9)). Their Appendix B evaluates the one-copy program exactly for antisymmetric supports P_- in dimension d, where $W_0(P_-)$ coincides with the min-Rains quantity and equals $d/(d+2)$. Wang and Duan had already evaluated W_0 exactly on mixed qutrit supports. Their family $\rho^{(\alpha)} = \sum_m |\psi_m\rangle\langle\psi_m|/3$ with $|\psi_0\rangle = \sqrt{\alpha}|01\rangle + \sqrt{1-\alpha}|10\rangle$ and its analogues on $\{|02\rangle, |20\rangle\}$ and $\{|12\rangle, |21\rangle\}$ has a support built, like the support P studied here, from one vector in each subspace $\mathrm{span}\{|ij\rangle, |ji\rangle\}$; at $\alpha = 1/2$ they showed that the one-shot and asymptotic exact PPT rates both equal $\log_2(3/2)$, so that $W_0 = 2/3$ [4, Prop. 5 and the closing discussion]. For a different qutrit family, obtained by cyclic shifts of $\sqrt{\alpha}|00\rangle + \sqrt{1-\alpha}|11\rangle$, they proved $W_0 = 1-\alpha$ for $0 < \alpha \leq 1/5$ [5, Prop. 4 and Example]. In all these cases, as for pure states [7, Prop. 4], W_0 coincides with the min-Rains quantity. For the qutrit

support they do not compute $W_0(P)$, give no achievability bound on the rate, and do not use a symmetrization argument. The contribution of the present note is therefore the exact value of $W_0(P)$ on a support where W_0 differs from the min-Rains quantity, the explicit optimal effect, the resulting lower bound on the rate and the values of $M_1, \dots, M_4$. We found none of these in the works cited here; this records a documented search and does not certify priority.

3 Setting

Let $\{|0\rangle, |1\rangle, |2\rangle\}$ be the computational basis of $\mathbb{C}^3$ and write $|ij\rangle = |i\rangle \otimes |j\rangle$. The support of Zhu and Wang [8, Sec. 3] is the rank-three projector $P = \sum_{k=0}^{2} |u_k\rangle\langle u_k|$ with

$$|u_0\rangle = \frac{8|01\rangle + 7|10\rangle}{\sqrt{113}}, \qquad |u_1\rangle = \frac{7|12\rangle + 8|21\rangle}{\sqrt{113}}, \qquad |u_2\rangle = \frac{|20\rangle + i|02\rangle}{\sqrt{2}}. \tag{2}$$

For $i < j$ let $H_{ij} = \operatorname{span}\{|ij\rangle, |ji\rangle\}$; we call the three pairs $e \in \{01, 12, 02\}$ edges. In the ordered bases $(|ij\rangle, |ji\rangle)$ the projector P is block diagonal with blocks

$$P_{01} = \frac{1}{113}\begin{pmatrix} 64 & 56 \\ 56 & 49 \end{pmatrix}, \qquad P_{12} = \frac{1}{113}\begin{pmatrix} 49 & 56 \\ 56 & 64 \end{pmatrix}, \qquad P_{02} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \tag{3}$$

which are equations (3.6)–(3.8) of [8], and P vanishes on $\operatorname{span}\{|00\rangle, |11\rangle, |22\rangle\}$. Each block is a rank-one projector. We denote by p_e the larger diagonal entry of P_e and by c_e its off-diagonal entry in position $(|ij\rangle, |ji\rangle)$:

$$p_{01} = p_{12} = \tfrac{64}{113}, \quad p_{02} = \tfrac{1}{2}, \qquad c_{01} = c_{12} = \tfrac{56}{113}, \quad c_{02} = \tfrac{i}{2}.$$

The partial transpose acts on the second factor, $(|ij\rangle\langle kl|)^{\Gamma} = |il\rangle\langle kj|$. Transposing the first factor instead gives the full transpose of E^{Γ}, which has the same operator norm.

The quantity of interest is $W_0(P)$ in (1), and for n copies $W_0(P^{\otimes n})$, with the partial transpose taken over all of Bob's qutrits. For a state ρ with support P, let M_n be the largest M such that some PPT operation maps $\rho^{\otimes n}$ to Φ_M exactly. As recalled above, $M_n = \lfloor W_0(P^{\otimes n})^{-1} \rfloor$ [4, 8]; it depends on ρ only through P.

Lemma 1 (Fixed action [8, Lemma 2.1])**.** *If P is a projector and $P \le E \le \mathbf{1}$, then $EP = PE = P$. Consequently $E = P + QEQ$ with $Q = \mathbf{1} - P$ and $0 \le QEQ \le Q$.*

Proof. For $|v\rangle$ in the range of P, $1 = \langle v|P|v\rangle \le \langle v|E|v\rangle \le 1$, so $\langle v|(\mathbf{1} - E)|v\rangle = 0$. Since $\mathbf{1} - E \ge 0$, this forces $(\mathbf{1} - E)|v\rangle = 0$. Hence $EP = P$, and $PE = P$ by taking adjoints. □

4 The one-copy optimum

Theorem 2. *For the support (2),*

$$W_0(P) = \lambda_* = \frac{\sqrt{177}}{7 + \sqrt{177}} = \frac{177 - 7\sqrt{177}}{128} \approx 0.6552426338.$$

The minimum is attained by the effect E_* that is zero on $|00\rangle, |11\rangle, |22\rangle$ and equals $P_e + t_e(\mathbf{1} - P_e)$ on each H_e, with

$$t_e = \frac{\lambda_* - p_e}{1 - p_e}, \qquad \text{that is} \qquad \begin{aligned} t_{01} = t_{12} &= \frac{241}{128} - \frac{113\sqrt{177}}{896} \approx 0.20495, \\ t_{02} &= 2\lambda_* - 1 = \frac{113 - 7\sqrt{177}}{64} \approx 0.31049. \end{aligned}$$

In particular one copy of any state supported on P gives no exact Bell pair under PPT operations: $M_1 = \lfloor 1 + 7/\sqrt{177} \rfloor = 1$.

The proof has three steps: a symmetry reduction, a lower bound for symmetric effects, and the verification that E_* attains it.

Symmetry reduction. For real phases $\varphi = (\varphi_0, \varphi_1, \varphi_2)$ let $U_\varphi = \sum_k e^{i\varphi_k}|k\rangle\langle k|$. The vectors (2) are eigenvectors of $U_\varphi \otimes U_\varphi$, with eigenvalues $e^{i(\varphi_0+\varphi_1)}$, $e^{i(\varphi_1+\varphi_2)}$ and $e^{i(\varphi_0+\varphi_2)}$, so $(U_\varphi \otimes U_\varphi)P(U_\varphi \otimes U_\varphi)^\dagger = P$. If E is feasible, so is $E_\varphi = (U_\varphi \otimes U_\varphi)E(U_\varphi \otimes U_\varphi)^\dagger$. On a matrix unit, conjugation by $U_\varphi \otimes U_\varphi$ multiplies $|ij\rangle\langle kl|$ by $e^{i(\varphi_i+\varphi_j-\varphi_k-\varphi_l)}$, and conjugation by $U_\varphi \otimes \bar{U}_\varphi$ multiplies its partial transpose $|il\rangle\langle kj|$ by the same phase. Hence $E_\varphi^\Gamma = (U_\varphi \otimes \bar{U}_\varphi)E^\Gamma(U_\varphi \otimes \bar{U}_\varphi)^\dagger$ and $\|E_\varphi^\Gamma\|_\infty = \|E^\Gamma\|_\infty$. The average $\bar{E} = (2\pi)^{-3}\int E_\varphi \, d\varphi$ is feasible by convexity of the constraints, and by convexity of the norm $\|\bar{E}^\Gamma\|_\infty \le \|E^\Gamma\|_\infty$. The averaged effect keeps exactly the matrix elements $\langle ij|E|kl\rangle$ with $\{i,j\} = \{k,l\}$ as multisets, because only then is the character $\varphi_i + \varphi_j - \varphi_k - \varphi_l$ identically zero. It is therefore enough to minimize over effects that are diagonal on $|00\rangle, |11\rangle, |22\rangle$ and block diagonal on the three spaces H_e.

For such an E, Lemma 1 applied block by block gives $E|_{H_e} = P_e + t_e(\mathbf{1} - P_e)$ with $t_e \in [0,1]$, because $\mathbf{1} - P_e$ has rank one on H_e; on the remaining vectors $E|ii\rangle = d_i|ii\rangle$ with $d_i \in [0,1]$. Conversely every choice of $t_e, d_i \in [0,1]$ is feasible. The diagonal entries of $E|_{H_e}$ are $p_e + t_e(1-p_e)$ and $(1-p_e) + t_e p_e$, and its off-diagonal entry is $(1-t_e)c_e$.

Partial transpose of a symmetric effect. The partial transpose fixes the diagonal matrix units $|ij\rangle\langle ij|$ and sends $|ij\rangle\langle ji|$ to $|ii\rangle\langle jj|$. Thus E^Γ is diagonal on the six vectors $|ij\rangle$ with $i \ne j$, with the diagonal entries of the edge blocks, and has a 3×3 block $B = D + A$ on $|00\rangle, |11\rangle, |22\rangle$, where $D = \mathrm{diag}(d_0, d_1, d_2)$ and A is the Hermitian matrix with zero diagonal and

$$A_{ij} = \langle ii|E^\Gamma|jj\rangle = \langle ij|E|ji\rangle = (1 - t_{ij})\, c_{ij} \qquad (ij = 01, 12, 02).$$

The characteristic polynomial of a Hermitian 3×3 matrix with zero diagonal is $x^3 - s\,x - 2\,\mathrm{Re}(A_{01}A_{12}\overline{A_{02}})$ with $s = |A_{01}|^2 + |A_{12}|^2 + |A_{02}|^2$. Here c_{01}, c_{12} are real and c_{02} is imaginary, so the cyclic product is imaginary and the spectrum of A is $\{0, \pm\sqrt{s}\}$.

Lower bound. Let $\lambda = \|E^\Gamma\|_\infty$ for a symmetric feasible E. If $\lambda \ge 1$ there is nothing to prove, so assume $\lambda < 1$. The diagonal entries give $\lambda \ge p_e + t_e(1-p_e)$, that is

$$1 - t_e \ge \frac{1-\lambda}{1-p_e} > 0 \qquad (e = 01, 12, 02).$$

For the unit eigenvector $|v\rangle$ of A with eigenvalue $\sqrt{s}$, positivity of D gives $\lambda \ge \langle v|(D+A)|v\rangle \ge \sqrt{s}$. Therefore

$$\lambda^2 \ge \sum_e |c_e|^2(1-t_e)^2 \ge (1-\lambda)^2 \sum_e \frac{|c_e|^2}{(1-p_e)^2} = (1-\lambda)^2\Big(\tfrac{64}{49} + \tfrac{64}{49} + 1\Big) = (1-\lambda)^2\,\frac{177}{49},$$

using $|c_e|/(1-p_e) = \frac{56/113}{49/113} = \frac{8}{7}$ for $e = 01, 12$ and $\frac{1/2}{1/2} = 1$ for $e = 02$. Taking square roots, $7\lambda \ge (1-\lambda)\sqrt{177}$, i.e. $\lambda \ge \lambda_*$. By the symmetry reduction the same bound holds for every feasible E.

Attainment. Since $\lambda_* \approx 0.6552 > 64/113 \approx 0.5664$, the values t_e of the theorem lie in $(0,1)$, and E_* is feasible. Its larger diagonal entry on H_e is $p_e + t_e(1-p_e) = \lambda_*$, and the smaller one, $(1-p_e) + t_e p_e$, does not exceed it because $1 - p_e \le p_e$. With $d_i = 0$ the block B equals A, and $1 - t_e = (1-\lambda_*)/(1-p_e)$ gives $s = (1-\lambda_*)^2 \cdot 177/49 = \lambda_*^2$. The spectrum of E_*^Γ is therefore contained in $[-\lambda_*, \lambda_*]$ and contains $\pm\lambda_*$, so $\|E_*^\Gamma\|_\infty = \lambda_*$. The closed forms follow from $\lambda_* = \sqrt{177}(\sqrt{177} - 7)/(177 - 49) = (177 - 7\sqrt{177})/128$. Finally $\lambda_*^{-1} = 1 + 7/\sqrt{177} \approx 1.5262 < 2$, so $M_1 = 1$. □

Remark 3 (Dual witness)**.** Any density matrix Y certifies a lower bound without the symmetry reduction: by Lemma 1, every feasible E satisfies

$$\|E^\Gamma\|_\infty \geq \mathrm{Tr}(E^\Gamma Y) = \mathrm{Tr}(PY^\Gamma) + \mathrm{Tr}(QEQ\,Y^\Gamma) \geq \mathrm{Tr}(PY^\Gamma) - \mathrm{Tr}\big[(QY^\Gamma Q)_-\big],$$

where X_- is the negative part. Let $|\phi\rangle$ be the eigenvector of E_*^Γ for the eigenvalue λ_* (it lies in the span of $|00\rangle, |11\rangle, |22\rangle$) and

$$Y = (1-\lambda_*)|\phi\rangle\langle\phi| + \lambda_*\Big(\tfrac{64}{177}|01\rangle\langle 01| + \tfrac{64}{177}|21\rangle\langle 21| + \tfrac{49}{177}|02\rangle\langle 02|\Big).$$

The weights mirror the proof: 64/49, 64/49 and 1 are the squared ratios $|c_e|^2/(1-p_e)^2$. Numerically, the right-hand side equals λ_* to within 10^{-15}, an independent confirmation of the lower bound. We have not written out an exact evaluation of this certificate.

5 Several copies

Corollary 4. *For every state ρ with support P and every $n \geq 1$,*

$$\Big(1 + \frac{7}{\sqrt{177}}\Big)^n \leq W_0(P^{\otimes n})^{-1} < \Big(\frac{391}{250}\Big)^n.$$

Consequently $M_1 = 1$, $M_2 = 2$, $M_3 = 3$ and $M_4 = 5$, while $M_5 \in \{8, 9\}$. The asymptotic rate satisfies

$$\log_2\Big(1 + \frac{7}{\sqrt{177}}\Big) \leq E^\infty_{0,\mathrm{PPT}}(\rho) < \log_2 \frac{391}{250}, \qquad \textit{numerically} \quad 0.60990 \leq E^\infty_{0,\mathrm{PPT}}(\rho) < 0.64525.$$

Proof. The upper bound is the witness bound of Zhu and Wang, $W_0(P^{\otimes n}) \geq \|T^\Gamma\|_1^{-n}$ with $\|T^\Gamma\|_1 < 391/250$ [8, Theorem 2.2 and Eq. (3.9)]. For the lower bound, write $E_* = P + R$ with $R = QE_*Q \geq 0$ and $PR = 0$ (Lemma 1). Expanding $E_*^{\otimes n}$ gives $P^{\otimes n}$ plus a sum of tensor products of copies of P and R, each positive semidefinite, so $P^{\otimes n} \leq E_*^{\otimes n} \leq \mathbf{1}$. Since $(E_*^{\otimes n})^\Gamma = (E_*^\Gamma)^{\otimes n}$ and the operator norm is multiplicative on tensor products, $W_0(P^{\otimes n}) \leq \lambda_*^n$, and $\lambda_*^{-1} = 1 + 7/\sqrt{177}$.

For the integers, $\sqrt{177} < 13.305$ gives $1 + 7/\sqrt{177} > 1 + 7/13.305 > 1.52612$, and the two bounds become

n	lower bound λ_*^{-n}	upper bound $(391/250)^n$	M_n
1	1.5262	1.5640	1
2	2.3291	2.4461	2
3	3.5546	3.8257	3
4	5.4249	5.9834	5
5	8.2792	9.3580	8 or 9

Since M_n is an integer with $\lambda_*^{-n} \leq W_0(P^{\otimes n})^{-1}$, we have $M_n \geq \lfloor \lambda_*^{-n} \rfloor$, and $M_n < (391/250)^n$. For $n \leq 4$ the lower bound is at least M and the upper bound is at most $M+1$ for the value M listed in the table, which fixes $M_n = M$. The checker verifies these inequalities in rational arithmetic. For the rate, $\frac{1}{n}\log_2 M_n \geq \frac{1}{n}\log_2 \lfloor \lambda_*^{-n} \rfloor \to \log_2 \lambda_*^{-1}$, and the upper bound is that of [8]. □

Two features of these numbers deserve comment. First, one copy yields nothing while two copies yield an exact Bell pair; the gain comes entirely from the tensor structure, since $E_*^{\otimes 2}$ has norm $\lambda_*^2 \approx 0.4293 < 1/2$. Second, the exact output is superadditive in the number of

copies: $M_3 = 3 > M_1 M_2 = 2$ and $M_4 = 5 > M_2^2 = 4$, so four copies processed jointly give a five-dimensional maximally entangled state, whereas two independent two-copy runs give dimension four. These statements hold for every state supported on P, because W_0 depends only on the support.

The lower bound $\log_2(1 + 7/\sqrt{177}) \approx 0.6099$ on the rate comes from tensor powers of the one-copy optimum. If W_0 were multiplicative on this support, it would be the exact rate; the witness value $\log_2(391/250) \approx 0.6452$ would be the exact rate if the witness were attained. The gap between them, about 0.035 bits per copy, is what the present results leave open. Both values lie below the min-Rains lower bound $-\log_2(6393/10000) \approx 0.6454$ of [8].

6 Verification

The theorem and the corollary are proved above. The public Python checker and its recorded output complement the proofs in four steps. With exact symbolic arithmetic (SymPy) it builds P from (2), checks the blocks (3), builds E_*, checks feasibility, and verifies that E_*^Γ has diagonal entries at most λ_* and that its 3×3 block has characteristic polynomial $x(x^2 - \lambda_*^2)$. In floating point it then tests the lower bound in three ways: 20 000 random feasible effects $E = P + QXQ$ with $0 \le X \le \mathbf{1}$ (smallest norm found ≈ 0.779), a multistart minimization over the six-parameter symmetric family (minimum λ_* to machine precision), and the dual witness of Remark 3. The first and third tests do not use the symmetry reduction. Finally it verifies the integer bounds of Corollary 4 with rational numbers. No input certificate is needed: the support is given by the rational data (2).

7 Scope and limitations

The exact value of $W_0(P)$ and the optimal effect are proved analytically. The values $M_1, \ldots, M_4$ and the rate interval follow from that proof combined with the witness bound of Zhu and Wang, whose certified inequality $\|T^\Gamma\|_1 < 391/250$ we take from their paper and do not re-derive. The numerical tests are a consistency check, not part of the proof. The results concern this particular support; the method extends to supports with the same phase symmetry, where the reduced problem again involves a 3×3 block, but the cancellation of the cyclic product used here depends on the phase i in $|u_2\rangle$. Whether $W_0(P^{\otimes n}) = \lambda_*^n$ for all n, the value $M_5 \in \{8, 9\}$, and the exact asymptotic rate remain open. The bibliographic comparison of Section 2 is a targeted reading of the cited sources and does not establish priority.

References

[1] M. Horodecki, P. Horodecki and R. Horodecki, *Mixed-state entanglement and distillation: is there a "bound" entanglement in nature?*, Phys. Rev. Lett. **80**, 5239 (1998); doi:10.1103/PhysRevLett.80.5239; arXiv:quant-ph/9801069.

[2] E. M. Rains, *Bound on distillable entanglement*, Phys. Rev. A **60**, 179–184 (1999); doi:10.1103/PhysRevA.60.179; revised version *An improved bound on distillable entanglement*, arXiv:quant-ph/9809082 (2000).

[3] E. M. Rains, *A semidefinite program for distillable entanglement*, IEEE Trans. Inf. Theory **47**, 2921–2933 (2001); doi:10.1109/18.959270; arXiv:quant-ph/0008047.

[4] X. Wang and R. Duan, *Improved semidefinite programming upper bound on distillable entanglement*, Phys. Rev. A **94**, 050301(R) (2016); doi:10.1103/PhysRevA.94.050301; arXiv:1601.07940.

[5] X. Wang and R. Duan, *Nonadditivity of Rains' bound for distillable entanglement*, Phys. Rev. A **95**, 062322 (2017); doi:10.1103/PhysRevA.95.062322; arXiv:1605.00348.

[6] K. Fang, X. Wang, M. Tomamichel and R. Duan, *Non-asymptotic entanglement distillation*, IEEE Trans. Inf. Theory **65**, 6454–6465 (2019); doi:10.1109/TIT.2019.2914688; arXiv:1706.06221.

[7] W. Matthews and A. Winter, *Pure-state transformations and catalysis under operations that completely preserve positivity of partial transpose*, Phys. Rev. A **78**, 012317 (2008); doi:10.1103/PhysRevA.78.012317; arXiv:0801.4322.

[8] C. Zhu and X. Wang, *The min-Rains relative entropy is not tight for exact PPT entanglement distillation*, arXiv:2608.12135v1 (2026).