Spectra of particle-reduced fermionic density operators: the Amosov–Filippov conjecture holds up to a factor $p!$

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Abstract

The p-particle reduced density matrix of a many-fermion state keeps exactly the information needed to evaluate p-body observables, and for a pure state with a fixed number N of particles a classical duality, usually attributed to Carlson and Keller, says that keeping p particles and keeping $N-p$ particles produce the same nonzero spectrum. Amosov and Filippov asked what replaces this duality on the whole fermionic Fock space, where the particle number need not be fixed. They introduced the map $\Phi(\varrho) = \sum_j a_j \varrho a_j^\dagger$, which removes one particle, and conjectured that, for every pure state, the operator $\Phi^p(\varrho)$ obtained by removing p particles has the same spectrum as the p-particle reduced density operator ϱ_p. We settle the conjecture. For every pure state ϱ on the Fock space of n modes and every p, the full spectra satisfy $\operatorname{Spec} \Phi^p(\varrho) = p!\, \operatorname{Spec} \varrho_p$, zero eigenvalues included. The conjecture is therefore true as stated for $p = 1$ and false for every $p \geq 2$ unless both operators vanish; the minimal counterexample is the two-mode state $|11\rangle$. The proof identifies ϱ_p and $\Phi^p(\varrho)/p!$ as the two complementary outputs of the completely positive map with Kraus operators $a_{s_p} \cdots a_{s_1}$. The same factor corrects the fixed-particle-number identity of the source, which becomes $\Phi^{N-p}(\varrho) = (N-p)!\, \varrho_p$.

1 Introduction

A state of n fermionic modes is a density operator on the Fock space H_n, of dimension 2^n. Many physical quantities depend on such a state only through a few particles at a time: the kinetic energy is a one-body observable, a pair interaction is a two-body observable. The object that carries exactly this information is the p-particle reduced density matrix, the central object of reduced-density-matrix methods in quantum chemistry. For a pure state of exactly N fermions, the p-particle and $(N-p)$-particle reduced density matrices have the same nonzero eigenvalues, a classical result that Amosov and Filippov attribute to Carlson and Keller [1] and that Gigena, Di Tullio and Rossignoli derive from a singular value decomposition [2, Sec. II B]. This is the fermionic analogue of the familiar fact that the two halves of a bipartite pure state have the same spectrum: removing $N-p$ particles and keeping p of them are spectrally equivalent operations.

States of fermions need not have a fixed particle number. Superconducting and other quasi-free states mix particle-number sectors, and the parity superselection rule only forbids superpositions of even and odd numbers. Amosov and Filippov [1] looked for the analogue of the Carlson–Keller duality on the whole Fock space. They introduced the linear map $\Phi(\varrho) = \sum_{j=1}^{n} a_j \varrho a_j^\dagger$, which removes one particle from any operator, proved that $\Phi^{N-p}(\varrho)$ equals the p-particle reduced density operator ϱ_p when the particle number is fixed, and conjectured that for every pure state on H_n, with no assumption on particle number or parity,

$\Phi^p(\varrho)$ and ϱ_p have the same spectrum. They checked the case $p = 1$, $n = 2$ by hand and reported that other examples, general pure states of three and four modes, are also equispectral.

The question answered here is whether this conjecture holds, and what the correct statement is if it does not. The answer is a single identity. For every pure state and every p,

$$\operatorname{Spec} \Phi^p(\varrho) = p!\ \operatorname{Spec} \varrho_p, \tag{1}$$

for the full spectra of the two operators on H_n, counted with multiplicity. The conjecture is thus true for $p = 1$ and false for every $p \geq 2$ except in the trivial case where both operators vanish. The factor $p!$ counts the orderings in which Φ^p removes the same set of p particles, and it is already visible in the traces: $\operatorname{Tr} \varrho_p$ is the expected number of p-element subsets of occupied modes, whereas $\operatorname{Tr} \Phi^p(\varrho)$ is the expected number of ordered p-tuples. The simplest example shows everything. For two modes and the state $|11\rangle$ in which both are occupied, $\varrho_2 = |11\rangle\langle 11|$ has spectrum $\{1, 0, 0, 0\}$, while removing both particles one after the other in either order gives $\Phi^2(\varrho) = 2|00\rangle\langle 00|$, with spectrum $\{2, 0, 0, 0\}$. This state has a fixed particle number $N = 2$, so the corollary that the source derives for fixed particle number also requires the factor.

Behind (1) lies a structural fact. The p-particle reduced density operator and $\Phi^p(\varrho)/p!$ are the two complementary outputs, in the sense of a Stinespring dilation, of one completely positive map whose Kraus operators are the ordered products of p annihilation operators. Two complementary outputs of a completely positive map acting on a pure input always share their nonzero spectrum, exactly as the two marginals of a bipartite pure state do. The particle-number-free extension sought by Amosov and Filippov therefore exists, with $\Phi^p(\varrho)/p!$ in the role of the $(N - p)$-particle reduced density operator when N is no longer defined.

Section 2 relates the result to the literature. Section 3 recalls the definitions of the source. Section 4 proves the main theorem and its consequences for the conjecture and for the fixed-particle-number proposition of the source. Section 5 describes the numerical checks and Section 6 the scope of the result.

2 Related work

The spectral duality between the p-particle and $(N - p)$-particle reduced density matrices of a pure N-fermion state is classical; Amosov and Filippov attribute it to Carlson and Keller and use it to derive the corollary of their Proposition 4 [1]. It is the fixed-N special case of Theorem 5 combined with Corollary 9 below.

Amosov and Filippov [1] define the reduced density operator ϱ_p on the full Fock space, introduce the particle-removing map Φ, and study spectral properties of reductions under the parity superselection rule. Their Proposition 4 states $\Phi^{N-p}(\varrho) = \varrho_p$ for states with fixed particle number N, and the corollary that follows it deduces equality of the spectra of ϱ_p and $\Phi^p(\varrho)$ for pure fixed-N states. The conjecture treated here closes their Section 5. We prove that Proposition 4 holds exactly for $N-p \leq 1$ and requires the factor $(N-p)!$ otherwise. Their proof correctly shows that Φ maps an m-particle operator to its reduction by one particle; iterating this relation counts each ordered removal separately, which produces the factorial.

Gigena, Di Tullio and Rossignoli [2] study many-body entanglement in fermion systems through M-body density operators. For an N-fermion state they write $\hat{\rho}^{(M)} = \frac{1}{(N-M)!} \sum_{j_1,\ldots,j_{N-M}} c_{j_1} \cdots c_{j_{N-M}}\, \rho\, c^\dagger_{j_{N-M}} \cdots c^\dagger_{j_1}$ [2, Eq. (79)], normalized so that $\operatorname{Tr} \rho^{(M)} = \binom{N}{M}$ [2, Eq. (16)], and they show that, for a pure state, $\rho^{(M)}$ and $\rho^{(N-M)}$ have the same spectrum apart from the number of zero eigenvalues [2, Sec. II B]. Their prefactor $1/(N - M)!$ is

exactly the correction of Proposition 4 established in Corollary 9; they refer to Amosov and Filippov for the extension to states without fixed particle number, which they do not treat, and they do not comment on the discrepancy with Proposition 4. The present note supplies that extension.

3 Setting and the conjecture

The Fock space H_n of n fermionic modes has the occupation basis $|j_1 \dots j_n\rangle$, $j_s \in \{0,1\}$. Following [1], the creation operators act by

$$a_t^\dagger |j_1 \dots j_n\rangle = (-1)^{j_1+\cdots+j_{t-1}} |j_1 \dots j_{t-1}\, 1\, j_{t+1} \dots j_n\rangle \quad \text{if } j_t = 0,$$

and $a_t^\dagger |j_1 \dots j_n\rangle = 0$ if $j_t = 1$; the annihilation operator a_t is the adjoint of $a_t^\dagger$. These operators satisfy the canonical anticommutation relations. The particle-number operator is $\hat{N} = \sum_s a_s^\dagger a_s$. For a density operator ϱ on H_n we write $\langle X \rangle = \mathrm{Tr}(\varrho X)$.

For a subset $S = \{s_1 < \cdots < s_p\}$ of $\{1, \dots, n\}$ we set $A_S = a_{s_p} \cdots a_{s_1}$, so that $A_S^\dagger = a_{s_1}^\dagger \cdots a_{s_p}^\dagger$, and $|S\rangle = A_S^\dagger |0\rangle$, where $|0\rangle$ is the vacuum. With the sign convention above, $|S\rangle$ is the occupation vector whose occupied modes are those of S, with sign $+1$, because each $a_{s_q}^\dagger$ acts on a vector whose modes of index below s_q are empty. The vectors $|S\rangle$ with $|S| = p$ form an orthonormal basis of the p-particle sector.

Definition 1 ([1])**.** The p-particle reduced density operator of ϱ is the operator on H_n

$$\varrho_p = \sum_{\substack{s_1<\cdots<s_p \\ t_1<\cdots<t_p}} \langle a_{s_1}^\dagger \cdots a_{s_p}^\dagger a_{t_p} \cdots a_{t_1} \rangle \; a_{t_1}^\dagger \cdots a_{t_p}^\dagger a_{s_p} \cdots a_{s_1} \prod_{s \notin \{s_1,\dots,s_p,t_1,\dots,t_p\}} a_s a_s^\dagger .$$

In the notation above, $\varrho_p = \sum_{|S|=|T|=p} \langle A_S^\dagger A_T \rangle \, E_{T,S}$ with $E_{T,S} = A_T^\dagger A_S \prod_{s \notin S \cup T} a_s a_s^\dagger$. No normalization is applied, so $\mathrm{Tr}\, \varrho_p$ need not equal 1.

The particle-removing map is $\Phi(X) = \sum_{j=1}^n a_j X a_j^\dagger$, and Φ^p denotes its p-fold composition.

Conjecture 2 (Amosov–Filippov [1])**.** *For every pure state $\varrho = |\psi\rangle\langle\psi|$ on H_n, with no assumption on particle number or parity, the operators ϱ_p and $\Phi^p(\varrho)$ have the same spectrum.*

4 Results

4.1 Two lemmas

The first lemma shows that the product of projectors in Definition 1 turns each term into a matrix unit of the p-particle sector.

Lemma 3. *For $|S| = |T| = p$, $E_{T,S} = |T\rangle\langle S|$. Consequently ϱ_p is supported on the p-particle sector, where $\langle T|\varrho_p|S\rangle = \mathrm{Tr}(\varrho A_S^\dagger A_T)$, and it vanishes on all other sectors.*

Proof. Let $|J\rangle$ be an occupation vector with set of occupied modes J. The product $\prod_{s \notin S\cup T} a_s a_s^\dagger$ is the projector onto vectors whose modes outside $S \cup T$ are empty, so it fixes $|J\rangle$ if $J \subseteq S \cup T$ and annihilates it otherwise. Next, $A_S|J\rangle$ vanishes unless $S \subseteq J$, in which case it equals $\pm|J \setminus S\rangle$. Finally, $A_T^\dagger |J \setminus S\rangle$ vanishes as soon as T meets $J \setminus S$; since $J \setminus S \subseteq T$, only $J = S$ survives. For $J = S$ we have $A_S|S\rangle = A_S A_S^\dagger |0\rangle = |0\rangle$, because each factor $a_s a_s^\dagger$ that appears acts on a vector in which mode s is empty, and $A_T^\dagger |0\rangle = |T\rangle$. Hence $E_{T,S}|J\rangle = \delta_{J,S}|T\rangle$. □

Lemma 4. *For every operator X on H_n and $1 \le p \le n$, $\Phi^p(X) = p! \sum_{|T|=p} A_T X A_T^\dagger$.*

Proof. By definition $\Phi^p(X) = \sum_{(j_1,\dots,j_p)} a_{j_p} \cdots a_{j_1} X a_{j_1}^\dagger \cdots a_{j_p}^\dagger$, the sum running over all p-tuples of mode indices. A tuple with a repeated index contributes zero, because the a_j anticommute and $a_j^2 = 0$. For a tuple of distinct indices with underlying set T, $a_{j_p} \cdots a_{j_1} = \pm A_T$, and the sign cancels against that of the adjoint. Each set T arises from $p!$ tuples. □

4.2 The main theorem

Theorem 5. *Let $|\psi\rangle \in H_n$ be a unit vector, $\varrho = |\psi\rangle\langle\psi|$ and $1 \le p \le n$. Let $V : \mathbb{C}^{\binom{n}{p}} \to H_n$ be the linear map defined on the standard basis $(e_T)_{|T|=p}$ by $V e_T = A_T|\psi\rangle$. Then*

$$\Phi^p(\varrho) = p!\, VV^\dagger, \qquad \langle T|\varrho_p|S\rangle = (V^\dagger V)_{S,T} \quad (|S| = |T| = p),$$

and the full spectra of the two operators on H_n, counted with multiplicity and zero eigenvalues included, satisfy $\operatorname{Spec} \Phi^p(\varrho) = p!\, \operatorname{Spec} \varrho_p$. Moreover, if $c = \sum_T c_T |T\rangle$ is an eigenvector of ϱ_p with eigenvalue $\lambda > 0$, then $\sum_T \overline{c_T}\, A_T|\psi\rangle$ is a nonzero eigenvector of $\Phi^p(\varrho)$ with eigenvalue $p!\,\lambda$.

Proof. Lemma 4 gives $\Phi^p(\varrho) = p! \sum_T A_T|\psi\rangle\langle\psi|A_T^\dagger = p!\, VV^\dagger$. Lemma 3 gives $\langle T|\varrho_p|S\rangle = \langle\psi|A_S^\dagger A_T|\psi\rangle = \langle A_S\psi|A_T\psi\rangle = (V^\dagger V)_{S,T}$. In the basis $(|T\rangle)$ of the p-particle sector, the matrix of ϱ_p is therefore the transpose of the Hermitian matrix $V^\dagger V$, that is, its complex conjugate. A Hermitian matrix and its complex conjugate have the same spectrum, and $VV^\dagger$ and $V^\dagger V$ have the same nonzero eigenvalues with the same multiplicities. The nonzero eigenvalues of $\Phi^p(\varrho)$ are thus those of ϱ_p multiplied by $p!$. Both operators act on the same space of dimension 2^n and have the same rank, so the multiplicities of the eigenvalue 0 agree as well. For the last statement, $\varrho_p c = \lambda c$ reads $\overline{V^\dagger V}\, c = \lambda c$, that is, $V^\dagger V \bar{c} = \lambda \bar{c}$. Then $VV^\dagger(V\bar{c}) = \lambda V\bar{c}$ and $\|V\bar{c}\|^2 = \lambda\|c\|^2 > 0$. □

The theorem has a transparent structure. The map $\mathcal{N}_p(X) = \sum_{|T|=p} A_T X A_T^\dagger$ is completely positive with Kraus operators A_T, and Lemma 3 identifies ϱ_p with the output of the complementary map: in the basis $(|T\rangle)_{|T|=p}$ of the p-particle sector, $\langle T|\varrho_p|S\rangle = \operatorname{Tr}(A_T \varrho A_S^\dagger)$, which is the matrix element of the complementary output indexed by the pair of Kraus operators (A_T, A_S). For a pure input, the output of a completely positive map and the output of its complement share their nonzero spectrum. The conjecture is therefore an instance of this general principle, and the factor $p!$ counts the orderings in which Φ^p removes the same set of p particles.

4.3 Consequences for the conjecture

Corollary 6 ($p = 1$)**.** *For every pure state on H_n, $\operatorname{Spec} \Phi(\varrho) = \operatorname{Spec} \varrho_1$. The conjecture holds for $p = 1$ with no assumption on parity or particle number.*

Corollary 7 ($p \ge 2$)**.** *Let $p \ge 2$ and $\varrho = |\psi\rangle\langle\psi|$. The spectra of $\Phi^p(\varrho)$ and ϱ_p coincide if and only if both operators vanish, that is, if and only if $|\psi\rangle$ has no component in the sectors with p or more particles. In particular the conjecture fails for every $p \ge 2$, already for the two-mode state $|11\rangle$ with $p = 2$, for which $\operatorname{Spec} \varrho_2 = \{1, 0, 0, 0\}$ and $\operatorname{Spec} \Phi^2(\varrho) = \{2, 0, 0, 0\}$.*

Proof. A multiset σ of nonnegative reals equals $p!\,\sigma$ with $p! > 1$ only if every element of σ is zero. By Theorem 5, both operators vanish exactly when $V = 0$, that is, when $A_T|\psi\rangle = 0$ for every $|T| = p$. Writing $|\psi\rangle = \sum_J \psi_J |J\rangle$, the vector $A_T|\psi\rangle$ is $\sum_{J \supseteq T} \pm\psi_J |J \setminus T\rangle$, and the vectors

$|J \setminus T\rangle$ are distinct for distinct $J \supseteq T$. Hence $V = 0$ if and only if $\psi_J = 0$ whenever $|J| \geq p$. For $|11\rangle$ we have $\varrho_2 = |11\rangle\langle 11|$ by Lemma 3, and $\Phi^2(\varrho) = 2A_{\{1,2\}}|11\rangle\langle 11|A^\dagger_{\{1,2\}} = 2|00\rangle\langle 00|$ by Lemma 4. □

The obstruction is already visible on the traces. Since $\operatorname{Tr}\varrho_p = \sum_{|S|=p}\langle A_S^\dagger A_S\rangle$ and $\sum_{|S|=p} A_S^\dagger A_S = \binom{\hat{N}}{p}$, while $\operatorname{Tr}\Phi^p(\varrho) = \sum_{(j_1,\dots,j_p)}\langle a^\dagger_{j_1}\cdots a^\dagger_{j_p}a_{j_p}\cdots a_{j_1}\rangle$ counts ordered tuples of distinct occupied modes,

$$\operatorname{Tr}\Phi^p(\varrho) = \langle \hat{N}(\hat{N}-1)\cdots(\hat{N}-p+1)\rangle = p!\left\langle \binom{\hat{N}}{p}\right\rangle = p!\,\operatorname{Tr}\varrho_p$$

for every state, pure or mixed.

The verdict depends on what one calls the p-particle reduced density matrix. The abstract of [1] speaks of the conventional p-particle reduced density matrix, and the source also writes it as the kernel indexed by ordered tuples. That reading is equispectral with $\Phi^p(\varrho)$ on the nonzero part.

Corollary 8 (tuple-indexed kernel)**.** *Let $D^{(p)}$ be the $n^p \times n^p$ matrix indexed by ordered p-tuples, $D^{(p)}_{s,t} = \langle a^\dagger_{s_1}\cdots a^\dagger_{s_p}a_{t_p}\cdots a_{t_1}\rangle$. For every pure state, $D^{(p)}$ and $\Phi^p(\varrho)$ have the same nonzero eigenvalues with the same multiplicities.*

Proof. Let W map the basis vector $e_{(t_1,\dots,t_p)}$ of $\mathbb{C}^{n^p}$ to $a_{t_p}\cdots a_{t_1}|\psi\rangle$. Then $D^{(p)} = W^\dagger W$, and the first line of the proof of Lemma 4 reads $\Phi^p(\varrho) = WW^\dagger$. □

The multiplicities of the eigenvalue 0 differ, since $n^p \neq 2^n$ in general. The conjecture is therefore exact on nonzero eigenvalues if the reduced density matrix is the tuple-indexed kernel, and false by the factor $p!$ for the operator ϱ_p of Definition 1, which is the operator for which the conjecture is stated in the body of [1]. In both readings the mathematical content is the identity (1).

4.4 Fixed particle number

Corollary 9 (correction of Proposition 4 of [1])**.** *If ϱ, pure or mixed, is supported on the N-particle sector and $0 \leq p \leq N$, then $\Phi^{N-p}(\varrho) = (N-p)!\,\varrho_p$, where $\varrho_0 = \operatorname{Tr}(\varrho)|0\rangle\langle 0|$.*

Proof. By linearity it suffices to treat $\varrho = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ in the N-particle sector. Put $m = N - p$. By Lemma 4 it suffices to prove $\sum_{|T|=m} A_T\varrho A_T^\dagger = \varrho_p$; for $m = 0$ this reads $\varrho = \varrho_N$, which is Lemma 3, and for $p = 0$ both sides equal $|0\rangle\langle 0|$. Both sides are supported on the p-particle sector. For $|J| = |K| = p$, Lemma 3 and the completeness of the basis $\{|T\rangle : |T| = m\}$ of the m-particle sector, which contains $A_J|\psi\rangle$ and $A_K|\psi\rangle$, give

$$\langle J|\varrho_p|K\rangle = \langle A_K\psi|A_J\psi\rangle = \sum_{|T|=m}\langle A_K\psi|T\rangle\langle T|A_J\psi\rangle.$$

Now $\langle T|A_J\psi\rangle = \langle 0|A_TA_J|\psi\rangle$ and $\langle J|A_T\psi\rangle = \langle 0|A_JA_T|\psi\rangle$, and $A_JA_T = (-1)^{pm}A_TA_J$ because annihilation operators anticommute. Hence $\langle T|A_J\psi\rangle = (-1)^{pm}\langle J|A_T\psi\rangle$, likewise for K, and the two signs cancel: $\langle J|\varrho_p|K\rangle = \sum_T\langle J|A_T\psi\rangle\langle A_T\psi|K\rangle = \langle J|(\sum_T A_T\varrho A_T^\dagger)|K\rangle$. □

Proposition 4 of [1] asserts $\Phi^{N-p}(\varrho) = \varrho_p$; it is exact for $N - p \leq 1$ and off by the factor $(N-p)!$ beyond. The corollary following it, equality of the spectra of ϱ_p and $\Phi^p(\varrho)$ for pure fixed-N states, is therefore also exact only for $p \leq 1$, as the example $|11\rangle$ shows. Combined with Theorem 5, Corollary 9 recovers the Carlson–Keller duality in the normalization of the

source: for a pure N-particle state, $\operatorname{Spec}\varrho_{N-p} = \frac{1}{p!}\operatorname{Spec}\Phi^p(\varrho) = \operatorname{Spec}\varrho_p$. Theorem 5 is thus the extension of this duality to the whole Fock space, with $\Phi^p(\varrho)/p!$ playing the role of the $(N-p)$-particle reduced density operator.

4.5 Mixed states

The spectral identity does not survive mixing, even for $p = 1$ and fixed particle number. For two modes and $\varrho = \frac{1}{2}(|10\rangle\langle 10| + |01\rangle\langle 01|)$, the operator ϱ_1 has spectrum $\{\frac{1}{2}, \frac{1}{2}, 0, 0\}$ while $\Phi(\varrho) = |00\rangle\langle 00|$ has spectrum $\{1, 0, 0, 0\}$. For mixed states only the trace identity $\operatorname{Tr}\Phi^p(\varrho) = p!\operatorname{Tr}\varrho_p$ and the complementarity relation survive in general. The source states the conjecture for pure states only.

5 Verification

The Python checker builds the ladder operators with the sign convention of [1], implements ϱ_p literally from Definition 1, projector product included, and Φ^p by composition, and writes its recorded output. It checks Lemma 3 for $n \le 4$; Lemma 4 on random mixed operators for $n \le 5$; the identity $\operatorname{Spec}\Phi^p(\varrho) = p!\operatorname{Spec}\varrho_p$ on all 2^n eigenvalues for 106 random pure states, real and complex, covering generic Fock-space states, even- and odd-parity states and fixed-N states with $N = 2, 3$ on four to six modes, with largest deviation of order 10^{-15}; the eigenvector map of Theorem 5; Corollary 8; Corollary 9 for pure and mixed states; the trace identities; the two exact counterexamples above; and the two-mode example of the source. For $p \ge 2$ the largest observed gap between the spectra of $\Phi^p(\varrho)$ and ϱ_p is about 5.29, while for $p = 1$ it is below 10^{-15}. The checker needs only NumPy and runs in a few seconds.

6 Scope and limitations

The analytic proof is complete and elementary. It covers every pure state on the Fock space of n modes, for all n and p, and uses only the canonical anticommutation relations and the equality of the nonzero spectra of $VV^\dagger$ and $V^\dagger V$. The numerical computations serve to confirm the sign and normalization conventions of the source; they are restricted to $n \le 6$ and are not a formal verification.

On bibliographic priority, the factorial correction at fixed particle number is not new: it appears explicitly, without comment on Proposition 4 of the source, in the definition of the M-body density operator of Gigena, Di Tullio and Rossignoli [2, Eq. (79)], and the fixed-N duality is classical. The extension to variable particle number, the identification of ϱ_p and $\Phi^p(\varrho)/p!$ as complementary outputs, and the observation that Proposition 4 and its corollary in [1] fail for $N - p \ge 2$ and $p \ge 2$ were not found in our search, which covered the 42 works citing [1] listed by Semantic Scholar in November 2026, with the citation contexts of the eight closest ones read at source, and keyword web searches. The result is elementary enough that it may appear elsewhere in another form; the search did not cover Google Scholar or quantum-chemistry monographs.

The generalized Pauli constraints for states obeying the parity superselection rule, raised in [1], remain outside the scope of this note, as does the spectrum of $\Phi^p(\varrho)$ for mixed states, where only the trace identity and the complementarity relation are available.

References

[1] G. G. Amosov and S. N. Filippov, *Spectral properties of reduced fermionic density operators and parity superselection rule*, Quantum Inf. Process. **16**, 2 (2017), doi:10.1007/s11128-016-1467-9, arXiv:1512.01828.

[2] N. Gigena, M. Di Tullio and R. Rossignoli, *Many-body entanglement in fermion systems*, Phys. Rev. A **103**, 052424 (2021), doi:10.1103/PhysRevA.103.052424, arXiv:2012.13785.