How many incoherent Kraus operators? A generic lower bound in every dimension and thirteen for the qutrit

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Abstract

In the resource theory of quantum coherence, the free operations are quantum channels that can be written with Kraus operators each of which maps incoherent (diagonal) states to incoherent states. A channel on a d-dimensional system always has a Kraus decomposition with at most d^2 operators, but the shortest decomposition of this restricted form can be longer, and the minimal number of incoherent Kraus operators is known only for a qubit, where Rana and Lewenstein showed that it equals four. We recast the question in graph language, as a decomposition of the Choi matrix into positive blocks supported on cliques of a complete multipartite graph; this restates the classification of incoherent operators by type due to Streltsov, Rana, Boes and Eisert. For a qubit the graph is a four-cycle, and a short fixed-point argument re-derives the value four. In dimension d, a dimension count shows that a generic incoherent completely positive map needs at least $\lceil d^2(d^2-d+1)/(2d-1)\rceil$ incoherent Kraus operators, which is 4, 13 and 30 for $d = 2, 3, 4$ and grows like $d^3/2$, faster than the largest Choi rank d^2. For the qutrit the bound 13 is attained: an exact rank certificate shows that thirteen incoherent operators of fixed shapes cover a non-empty open set of incoherent completely positive maps, and also a non-empty open set of trace-preserving incoherent operations. Whether generic trace-preserving qutrit operations need thirteen operators is left open; the best proven range for the worst qutrit operation remains 9 to 32.

1 Introduction

Coherence, the presence of superpositions in a fixed basis $\{|0\rangle, \ldots, |d-1\rangle\}$, is treated as a resource in the framework of Baumgratz, Cramer and Plenio [1]: states diagonal in the basis are free, and so are the incoherent operations, channels $\Lambda(\rho) = \sum_k K_k \rho K_k^\dagger$ that admit a Kraus decomposition in which every K_k maps diagonal density matrices to diagonal ones, up to normalization [1]. The definition depends on the decomposition: a channel may be incoherent through one set of Kraus operators and not through another. This raises a basic structural question. Every channel has a Kraus decomposition with at most d^2 operators, its Choi rank [6, Thm. 2 and Remark 4]. How many operators are needed if every one of them must be incoherent?

Streltsov, Rana, Boes and Eisert proved upper bounds on this number, showed that five incoherent Kraus operators suffice for a qubit, and asked whether four suffice [2]. Rana and Lewenstein answered the question: four incoherent Kraus operators suffice for every incoherent qubit channel, and four are optimal [4]. Qiao, Sun, Wang, Li, Shen, Li and Fei gave a second

proof of the qubit case and reduced the qutrit upper bound to 32 [5]. Rana and Lewenstein also showed that the incoherent length can exceed the Choi rank already for a qubit. For $d \geq 3$ no lower bound beyond the Choi rank d^2 appears in these works.

This note gives such a lower bound for incoherent completely positive maps, together with a matching construction for the qutrit. The main tool is a reformulation. An operator is incoherent exactly when each of its columns has at most one non-zero entry, so its Choi vector is supported on a set of matrix positions containing at most one position per column. Such sets are the cliques of a complete d-partite graph, and an incoherent Kraus decomposition is a decomposition of the Choi matrix into positive semidefinite blocks supported on cliques (Section 3). This is a graph-theoretic restatement of the Choi-matrix and type arguments of [2, 4]. For the qubit the graph is a four-cycle, and a monotone recurrence around the cycle yields a decomposition with four rank-one blocks; this re-derives the theorem of Rana and Lewenstein (Section 4). In general dimension, each incoherent Kraus operator carries $2d-1$ real parameters, whereas the incoherent Choi matrices span a space of dimension $d^2(d^2-d+1)$. Hence a generic incoherent completely positive map needs at least

$$r_d = \left\lceil \frac{d^2(d^2-d+1)}{2d-1} \right\rceil$$

incoherent Kraus operators (Section 5). For $d = 3$ this is 13, and an exact integer rank computation shows that thirteen operators of suitable shapes parametrize an open set of incoherent completely positive maps and an open set of incoherent channels (Section 6). Individual operations whose incoherent or strictly incoherent length exceeds the Choi rank were known for $d = 2$ and $d = 3$ [4]; the new point is that from $d = 3$ on the gap holds on an open set of incoherent completely positive maps and grows like $d^3/2$.

2 Related work

Baumgratz, Cramer and Plenio introduced the incoherent operations through Kraus decompositions and the conditions for coherence measures [1]. Winter and Yang introduced the strictly incoherent operations, whose Kraus operators K and $K^\dagger$ are both incoherent, and characterized an incoherent Kraus operator by a function $j(i)$ that places at most one non-zero entry in each column, strict incoherence meaning that this function is one-to-one [3].

Streltsov, Rana, Boes and Eisert studied the structure of incoherent operations and proved the first bounds on the number of incoherent Kraus operators [2]. A Carathéodory argument on the Choi matrix gives at most d^4+1 operators (their Proposition 3). Grouping incoherent operators by their type, the position of their non-zero entries, gives at most $d(d^d-1)/(d-1)$, which is 39 for a qutrit (Proposition 5); the d^d types are exactly the maximal cliques of the graph G_d of Section 3, so Proposition 1 restates their classification in graph language. They proved that five operators suffice for a qubit (Proposition 4), bounded strictly incoherent operations by $\sum_{k=1}^{d} d!/(k-1)!$, which is 15 for a qutrit (Proposition 6), exhibited strictly incoherent operations that need d^2 operators (Proposition 7), and asked whether four operators suffice for every qubit incoherent operation.

Rana and Lewenstein settled the qubit question [4]. They decomposed the Choi matrix of an incoherent qubit channel into four positive semidefinite blocks of rank one, which shows that four incoherent Kraus operators always suffice, and proved that four are optimal (their Theorem 2). They also gave a qubit incoherent operation of Kraus rank three that needs four incoherent operators (Proposition 6), proved that for a strictly incoherent qubit operation the strictly incoherent length equals the Kraus rank (Proposition 5), and exhibited a qutrit strictly incoherent operation of Kraus rank five that needs six strictly incoherent operators; they left

the case $d \geq 3$ open. Qiao et al. gave a second proof of the qubit result by constructing a suitable unitary matrix, obtained the upper bound 32 for qutrit incoherent operations, improving the bound 39 of [2], and reduced the bound for strictly incoherent operations of a qutrit from 15 to 13 [5]. That last number concerns a different, smaller class of operations; its coincidence with our $r_3 = 13$ is not a consequence of either argument.

Our qubit theorem is therefore the theorem of Rana and Lewenstein. Their proof and ours both decompose the Choi matrix into four rank-one blocks supported on the four edges of the cycle. They determine the blocks through a quadratic equation, whereas here they come from a fixed point of a monotone recurrence around the cycle. We include the argument because it is short and because the same clique formulation underlies the results in higher dimension. The new results of this note are the generic lower bound r_d of Theorem 4 and the qutrit results of Section 6. The Choi matrix [6] and an elementary measure estimate are the only other tools used.

3 The clique formulation

Write $e_{ij} = |i\rangle\langle j|$ and identify an operator K on $\mathbb{C}^d$ with its vector of entries $|K\rangle\!\rangle = \sum_{i,j} K_{ij}\, e_{ij} \in \mathbb{C}^{d^2}$. The Choi matrix of a completely positive map with Kraus operators $\{K_k\}$ is $J = \sum_k |K_k\rangle\!\rangle\langle\!\langle K_k|$, with entries $J_{(i,j),(i',j')} = \sum_k K_{k,ij}\overline{K_{k,i'j'}}$; it does not depend on the decomposition [6, Thm. 2]. Trace preservation is the linear condition $S(J) = \mathbf{1}$, where

$$S(J)_{jl} = \sum_i J_{(i,l),(i,j)} = \Big(\sum_k K_k^\dagger K_k\Big)_{jl}. \tag{1}$$

An operator K maps every diagonal matrix to a diagonal matrix exactly when $K|j\rangle\langle j|K^\dagger$ is diagonal for each j, that is, when each column $K|j\rangle$ has at most one non-zero entry. Such a K has the form

$$K = \sum_j c_j |f(j)\rangle\langle j|$$

for a function $f : \{0,\dots,d-1\} \to \{0,\dots,d-1\}$, which we call the *type* of K, and coefficients $c_j \in \mathbb{C}$. Its support is a *transversal*: a set of positions e_{ij} containing at most one position in each column j.

Let G_d be the graph with vertex set $\{e_{ij}\}$ in which two distinct positions are adjacent when they lie in different columns. It is the complete d-partite graph whose parts are the columns, and its cliques are exactly the transversals.

Proposition 1. *A completely positive map with Choi matrix J admits a decomposition into incoherent Kraus operators if and only if $J = \sum_T B_T$, where T runs over the maximal cliques of G_d and each B_T is positive semidefinite with support in $T \times T$. The minimal number of incoherent Kraus operators, which we call the incoherent length $\ell(J)$, is the minimum of $\sum_T \operatorname{rank} B_T$ over such decompositions.*

Proof. Given incoherent Kraus operators, assign each to a maximal clique containing its support and let B_T be the sum of $|K_k\rangle\!\rangle\langle\!\langle K_k|$ over the operators assigned to T; then $\operatorname{rank} B_T$ is at most the number of those operators. Conversely, a positive semidefinite B_T supported on $T \times T$ is a sum of $\operatorname{rank} B_T$ rank-one terms $|v\rangle\!\rangle\langle\!\langle v|$ with v supported on T, and each such v is the Choi vector of an incoherent operator. □

Because two positions in the same column never share a clique, every incoherent Choi matrix satisfies $J_{(i,j),(i',j)} = 0$ for $i \neq i'$. Let V_d be the real vector space of Hermitian $d^2 \times d^2$ matrices

with this zero pattern and $\mathcal{C}_d \subset V_d$ the convex cone of Choi matrices of incoherent completely positive maps.

Lemma 2. *$\dim_{\mathbb{R}} V_d = D_d := d^2(d^2-d+1)$, and $\mathcal{C}_d$ has non-empty interior in V_d.*

Proof. V_d has d^2 real diagonal coordinates and one complex coordinate for each unordered pair of positions in different columns. There are $\binom{d}{2}d^2$ such pairs (two columns, then a row in each), so $\dim V_d = d^2 + 2\binom{d}{2}d^2 = d^2(d^2-d+1)$. Every matrix unit allowed by the pattern lies in the span of $\mathcal{C}_d$: a diagonal position is the Choi matrix of a single-entry operator, and for two positions in different columns, $|u\rangle\!\rangle\langle\!\langle u|$ with u equal to $e_{ij}+e_{i'j'}$ or $e_{ij}+ie_{i'j'}$ is in $\mathcal{C}_d$, so the real and imaginary parts of the off-diagonal entry are obtained by subtracting diagonal terms. A convex cone that spans V_d has non-empty interior in V_d. □

4 The qubit: four operators suffice

For $d=2$ the graph G_2 is the four-cycle $v_0 = e_{00}$, $v_1 = e_{11}$, $v_2 = e_{10}$, $v_3 = e_{01}$, in which edge k joins v_k and v_{k+1} (indices modulo 4); the two non-adjacent pairs (v_0, v_2) and (v_1, v_3) lie in the same column. The four edges are the four maximal transversals, corresponding to diagonal operators, operators with image $|1\rangle$, antidiagonal operators and operators with image $|0\rangle$. An incoherent Choi matrix has diagonal entries $d_k = J_{v_k v_k} \geq 0$, an entry $c_k = J_{v_k v_{k+1}}$ on each edge, and zeros elsewhere. Put $m_k = |c_k|^2$.

Theorem 3 (Rana and Lewenstein [4]; see also [5])**.** *Every incoherent completely positive map of a qubit, in particular every incoherent operation, admits a decomposition into four incoherent Kraus operators. Four are necessary for the depolarizing channel.*

Proof. By Proposition 1, $J = \sum_{k=0}^{3} B_k$ with B_k positive semidefinite and supported on edge k; single-vertex blocks have been absorbed into an adjacent edge block, which keeps it positive. Write $B_k = \begin{pmatrix} \alpha_k & c_k \\ \bar{c}_k & \beta_k \end{pmatrix}$ in the basis (v_k, v_{k+1}). Then $\alpha_k, \beta_k \geq 0$, $\alpha_k\beta_k \geq m_k$, and $d_{k+1} = \alpha_{k+1} + \beta_k$ for every k. A block is of rank at most one exactly when $\alpha_k\beta_k = m_k$, and a rank-one block $|w\rangle\!\rangle\langle\!\langle w|$ with $w = \sqrt{\alpha_k}\, v_k + (\bar{c}_k/\sqrt{\alpha_k})\, v_{k+1}$ is the Choi matrix of one incoherent operator.

All $m_k > 0$. Then all α_k, β_k are positive. For $a > 0$ let $f_k(a) = d_{k+1} - m_k/a$, a continuous, strictly increasing function. The constraints give $\alpha_{k+1} = d_{k+1} - \beta_k \leq f_k(\alpha_k)$. Define the sequence $a_0 = a$, $a_{k+1} = f_k(a_k)$, and $F(a) = a_4$, as long as the a_k stay positive. Starting from $a = \alpha_0$, induction and monotonicity give $a_k \geq \alpha_k > 0$ for $k = 1, 2, 3$ and $F(\alpha_0) \geq f_3(\alpha_3) \geq \alpha_0$. For $a \geq \alpha_0$ every a_k is at least its value at α_0, hence positive, so $G(a) = F(a) - a$ is continuous on $[\alpha_0, \infty)$, with $G(\alpha_0) \geq 0$ and $G(a) \leq d_0 - a \to -\infty$. By the intermediate value theorem there is $a^* \geq \alpha_0$ with $F(a^*) = a^*$. Setting $\alpha_k = a_k(a^*)$ and $\beta_k = m_k/\alpha_k$ satisfies every vertex equation, the last one because $a_4 = a^* = \alpha_0$, and makes all four blocks of rank one.

Some $m_k = 0$. Relabel the cycle so that $m_3 = 0$. Then B_3 is diagonal, and its two diagonal entries can be moved into B_2 and B_0; we may thus assume $\alpha_0 = d_0$ and $d_3 \geq \beta_2$, and work on the path $v_0v_1v_2v_3$. Define $a_0 = d_0$ and $a_{k+1} = d_{k+1} - m_k/a_k$ for $k = 0, 1$, with the convention $m_k/a_k = 0$ when $m_k = 0$. As before $a_k \geq \alpha_k$ for $k = 0, 1, 2$, and $\alpha_k > 0$ whenever $m_k > 0$, so the recurrence is defined. The three blocks with diagonal entries $(a_k, m_k/a_k)$ are of rank at most one, and the remainder $d_3 - m_2/a_2 \geq d_3 - m_2/\alpha_2 \geq d_3 - \beta_2 \geq 0$ at v_3 is a single-entry operator. This gives four operators.

The four Pauli matrices are incoherent, and $\{\mathbf{1}/2, X/2, Y/2, Z/2\}$ is a Kraus decomposition of the completely depolarizing channel, whose Choi matrix $\mathbf{1}/2$ has rank four; no decomposition with fewer than four operators exists. □

5 A generic lower bound in dimension d

Theorem 4. *For r with $(2d-1)\,r < D_d$, the set of $J \in \mathcal{C}_d$ with $\ell(J) \le r$ is closed and of Lebesgue measure zero in V_d, hence nowhere dense. Consequently a generic incoherent completely positive map on $\mathbb{C}^d$ requires at least*

$$r_d = \left\lceil \frac{d^2(d^2-d+1)}{2d-1} \right\rceil$$

incoherent Kraus operators. The first values are $r_2 = 4$, $r_3 = 13$, $r_4 = 30$, $r_5 = 59$, and $r_d = d^3/2 + O(d^2)$.

Proof. Fix types $f_1, \dots, f_r$ and consider the polynomial map $\Phi : (c_{k,j}) \in \mathbb{C}^{dr} \mapsto J = \sum_k |K_k\rangle\!\rangle\langle\!\langle K_k|$ with $K_k = \sum_j c_{k,j} |f_k(j)\rangle\langle j|$. Multiplying K_k by a phase does not change J, and a phase makes the first non-zero coefficient of K_k real. Hence Φ has the same image on the union A of the real linear subspaces $L_\sigma \subset \mathbb{C}^{dr}$, indexed by $\sigma \in \{0, \dots, d-1\}^r$, on which $c_{k,j} = 0$ for $j < \sigma_k$ and c_{k,σ_k} is real; each L_σ has real dimension at most $(2d-1)r$. Since $\operatorname{Tr} J = \sum_{k,j} |c_{k,j}|^2$, the maps with $\ell(J) \le r$, these types and $\operatorname{Tr} J \le t$ form the image under Φ of the compact set $A_t = \{c \in A : \|c\|^2 \le t\}$. This image is compact, and it has measure zero in $V_d \cong \mathbb{R}^{D_d}$: on each $L_\sigma \cap A_t$ the polynomial map Φ is Lipschitz with some constant L, and covering a cube of side s in $\mathbb{R}^m$, $m \le (2d-1)r < D_d$, by N^m cubes of side s/N covers its image by N^m balls of radius $L\sqrt{m}\,s/N$, whose total volume is $O(N^{m-D_d}) \to 0$. Taking the finite union over the d^{dr} choices of types, the set $\{\ell \le r\} \cap \{\operatorname{Tr} J \le t\}$ is compact and of measure zero for every t. A limit J of points of $\{\ell \le r\}$ lies in one of these compact sets with $t = \operatorname{Tr} J + 1$, so $\{\ell \le r\}$ is closed; a closed set of measure zero has empty interior, so it is nowhere dense. Since $\mathcal{C}_d$ is a closed convex cone with non-empty interior (Lemma 2), it is the closure of its interior, and the complement of $\{\ell \le r\}$ contains an open dense subset of $\mathcal{C}_d$. Applying this to the largest r with $(2d-1)r < D_d$, namely $r_d - 1$, gives the bound. The asymptotic form follows from $d^2(d^2-d+1)/(2d-1) = d^3/2 - d^2/4 + O(d)$. □

For $d = 2$ the bound equals the value of Theorem 3, which holds for every map. For $d = 3$ a generic incoherent map, whose Choi rank is 9, has no incoherent decomposition with 9, 10, 11 or 12 operators. The incoherent length thus exceeds the Choi rank on an open set of maps as soon as $d \ge 3$, and the gap grows like $d^3/2$ against d^2. For $d = 2$ the gap occurs only on a lower-dimensional set, such as the operation of Kraus rank three in [4, Prop. 6].

6 The qutrit: thirteen operators are attained

Write a type as the triple $(f(0), f(1), f(2))$ and consider the thirteen types

$$\begin{gathered}(1,0,1),\ (1,2,0),\ (1,2,2),\ (0,1,1),\ (2,0,2),\ (1,1,0),\ (2,1,1),\\ (0,0,1),\ (2,0,0),\ (0,1,0),\ (2,1,2),\ (0,2,2),\ (2,2,1).\end{gathered} \tag{2}$$

Theorem 5. *Let $\Phi : \mathbb{C}^{39} \to V_3$ be the map of the proof of Theorem 4 for the types (2). Its image contains a non-empty open subset $\mathcal{U}$ of V_3, contained in $\mathcal{C}_3$. Every $J \in \mathcal{U}$ has $\ell(J) \le 13$, and $\ell(J) = 13$ for all J in an open dense subset of $\mathcal{U}$.*

Proof. Regard Φ as a map from $\mathbb{R}^{78}$ (real and imaginary parts) to $V_3 \cong \mathbb{R}^{63}$. At a point $c^{(0)}$ with Gaussian-integer coefficients, recorded in the public certificate, the real Jacobian of Φ is an integer 81×78 matrix in the coordinates given by the diagonal entries and the real and imaginary parts of the upper triangle of J; eighteen of its rows, those of the same-column

positions, vanish identically. Its rank modulo the prime $p = 2^{31} - 1$ is 63. A non-zero minor modulo p is a non-zero integer, so the rank over $\mathbb{R}$ is $63 = \dim V_3$. Thus Φ is a submersion at $c^{(0)}$, and by the local submersion theorem its image contains an open neighbourhood $\mathcal{U}$ of $\Phi(c^{(0)})$ in V_3. Every element of the image is the Choi matrix of the thirteen incoherent operators that produce it, so $\mathcal{U} \subset \mathcal{C}_3$ and $\ell \leq 13$ on $\mathcal{U}$. By Theorem 4, $\ell \geq 13$ on an open dense subset of $\mathcal{C}_3$, whose intersection with $\mathcal{U}$ is open and dense in $\mathcal{U}$. □

Trace-preserving maps form the affine slice $\{S(J) = \mathbf{1}\}$ of V_3, where S is the linear map (1); it has dimension $63 - 9 = 54$. The same certificate shows that thirteen operators also suffice on an open set of incoherent channels.

Theorem 6. *There is a non-empty set of incoherent operations on a qutrit, open in the affine space $\{J \in V_3 : S(J) = \mathbf{1}\}$, each of which admits a decomposition into thirteen incoherent Kraus operators of the types (2).*

Proof. The certificate point $c^{(0)}$ is chosen so that $\sum_k K_k^\dagger K_k = \mathrm{diag}(s_0, s_1, s_2)$ is diagonal, with $(s_0, s_1, s_2) = (91, 209, 155)$. Let $\Delta = \mathrm{diag}(s_j^{-1/2})$ and $c^{(1)}_{k,j} = c^{(0)}_{k,j} s_j^{-1/2}$, so that the operators become $K_k\Delta$, which are incoherent of the same types and satisfy $\sum_k \Delta K_k^\dagger K_k \Delta = \mathbf{1}$. Right multiplication by Δ acts linearly and invertibly on the parameters and on V_3, and intertwines Φ; hence the Jacobian of Φ at $c^{(1)}$ also has rank 63, and its image is all of V_3.

The restriction of S to V_3 maps onto the nine-dimensional space of Hermitian 3×3 matrices: the composite of S with the integer Jacobian at $c^{(0)}$ has rank 9 modulo p, hence over $\mathbb{R}$. Consequently $\Psi = S \circ \Phi$ has surjective differential at $c^{(1)}$, the fibre $N = \Psi^{-1}(\mathbf{1})$ is a smooth manifold near $c^{(1)}$, and its tangent space is $\ker(S \circ d\Phi) = d\Phi^{-1}(\ker S)$. Because $d\Phi$ maps onto V_3, the differential of $\Phi|_N$ maps onto $\ker S \cap V_3$, which is the tangent space of the 54-dimensional slice. So $\Phi|_N$ is a submersion onto the slice at $c^{(1)}$, and its image contains an open neighbourhood of $\Phi(c^{(1)})$ in the slice. Every point of this neighbourhood is the Choi matrix of a trace-preserving map with thirteen incoherent Kraus operators. □

The matching lower bound for channels is not proved. Theorem 4 concerns the full cone, and the trace-preserving slice has measure zero in it, so the theorem does not apply directly. The natural reduction does not close the gap: right multiplication by invertible diagonal matrices is the only obvious symmetry that preserves incoherence of the operators, and it can normalize the diagonal of $\sum_k K_k^\dagger K_k$ but not its off-diagonal entries. A naive count gives the same threshold, since at a point where the trace-preservation constraint is a submersion the fibre has dimension $6r - 9$, of which r dimensions are phases, so r operators reach at most a $(5r - 9)$-dimensional part of the 54-dimensional slice, and $5r - 9 \geq 54$ requires $r \geq 13$. This count ignores the singular points of the fibre, where the dimension can jump.

Conjecture 7. *A generic incoherent operation on a qutrit requires thirteen incoherent Kraus operators; more generally, r_d incoherent Kraus operators are required by a generic incoherent operation on $\mathbb{C}^d$.*

7 Verification

The theorems are proved above; Theorems 5 and 6 rest on exact rank computations that the public bundle reproduces. The Python checker reads the qutrit certificate, which lists the thirteen types and the Gaussian-integer point $c^{(0)}$, and writes its recorded output. It checks that $\sum_k K_k^\dagger K_k = \mathrm{diag}(91, 209, 155)$ exactly, builds the integer 81×78 Jacobian, confirms that eighteen rows vanish, and computes its rank and the rank of its composite with S modulo

$2^{31} - 1$ by Gaussian elimination in Python integers, obtaining 63 and 9. In floating point it also confirms rank 63 at ten random points, compares the rank of the parametrization with the ceiling $\min\{(2d-1)r, D_d\}$ for random types with $d = 2, 3$, and tabulates r_d for $d \le 6$. For the qubit it applies the construction of Theorem 3 to 4000 random incoherent maps given by five to twelve incoherent operators (2466 with all edge coherences non-zero, 1534 with a vanishing one); the reconstructed four operators are incoherent and reproduce the Choi matrix to a relative error below 10^{-15}.

8 Scope and limitations

Theorem 3 is the theorem of Rana and Lewenstein [4], also proved by Qiao et al. [5], given here with a short fixed-point proof. Theorems 4, 5 and 6 are proved, the last two through an exact integer rank certificate. The lower bound of Theorem 4 is generic: it says nothing about specific maps on the exceptional measure-zero set, and it is proved for incoherent completely positive maps, not for trace-preserving ones, for which Conjecture 7 remains open. For the worst case over all qutrit incoherent operations, the proven range is therefore $9 \le \max \ell \le 32$: the lower end is the Choi rank of the completely depolarizing qutrit channel, which has the nine incoherent Kraus operators $X^a Z^b/3$ built from the cyclic shift X and the clock matrix Z (strictly incoherent operations of Choi rank d^2 are also given in [2, Prop. 7]), and the upper end is the bound of [5]. The conjecture would raise the lower end to 13. By Proposition 1 the exact value is a rank minimization over decompositions supported on the 27 maximal transversals of a qutrit.

On bibliographic priority, the qubit theorem is due to Rana and Lewenstein, individual operations with a gap between Choi rank and incoherent length were known in dimensions two and three [4], and the clique formulation restates the Choi-matrix and type arguments of [2]. The generic lower bound r_d and the qutrit results were not found in the documented search, which covered the works citing [2], [4] and [5] listed by OpenAlex as of 19 November 2026 and targeted searches on the number of incoherent Kraus operators; this does not establish priority.

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