Kirkwood–Dirac positive states of finite abelian Fourier pairs: mixtures of pure states exhaust them only for cyclic groups of prime-power order

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Abstract

Two incompatible quantum measurements have no joint probability distribution in general, but the Kirkwood–Dirac (KD) representation assigns to every state a table of complex quasiprobabilities with the correct marginals. When the two measurements are the position basis and the Fourier basis attached to a finite abelian group G, the states whose table is a genuine probability distribution form a convex set, and its pure elements are known: they are uniform superpositions over a coset of a subgroup, multiplied by a character. This note determines when these pure states generate the whole set. Every KD-positive state is a mixture of pure KD-positive states if and only if G is cyclic of prime-power order. The "if" part is a theorem of De Bièvre, Langrenez and Radchenko, who also found counterexamples for $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$; the "only if" part was open for all other groups. Our proof has three ingredients. Subgroup and quotient embeddings preserve exactly the trace-norm distance to the pure-state hull, which reduces the question to the groups $\mathbb{Z}_p \times \mathbb{Z}_q$ with p, q prime. On such a group the four product bases form a Clauser–Horne–Shimony–Holt scenario, and an explicit Bell-type witness is nonnegative on every pure KD-positive state and negative on an explicit KD-positive state for all pairs (p, q) outside a list of eight. One of the eight, $\mathbb{Z}_2 \times \mathbb{Z}_2$, is settled by hand; six are settled by explicit states with exact rational KD tables whose positive definiteness is certified in interval arithmetic; the last, $\mathbb{Z}_3 \times \mathbb{Z}_3$, by a state certified in rational arithmetic alone, at trace-norm distance at least $111/20000$ from the hull.

1 Introduction

Two incompatible quantum measurements do not generally admit a joint probability distribution. The Kirkwood–Dirac (KD) representation nevertheless assigns a table of quasiprobabilities to a state and an ordered pair of orthonormal bases. Its row and column sums are ordinary measurement probabilities, but individual entries may be negative or complex. When every entry is real and nonnegative, the table is a genuine probability distribution and the state is called KD-positive for that pair of bases. This condition depends on the chosen bases, and its failure is a resource: for multiqubit systems, for instance, KD nonpositivity is necessary for a quantum computational advantage [9]. Among all quasiprobability representations, the KD ones are singled out by the fact that they alone produce natural conditional expectations [11].

The pairs of bases studied here come from a finite abelian group G. The first basis is labelled by the elements of G and the second is its Fourier transform, labelled by the characters of G. For a cyclic group this is the discrete Fourier transform, and for $G = \mathbb{Z}_2^n$ it is the pair of computational and Hadamard bases of n qubits. For such a pair the pure KD-positive states are completely

known: each is the uniform superposition over a coset of a subgroup, multiplied by a character [1]. There are finitely many of them, and every mixture of them is KD-positive.

The geometric question is whether the converse holds: is every KD-positive mixed state a mixture of pure KD-positive states? An affirmative answer makes the positive region a polytope whose vertices are listed by the pure-state classification, and several arguments about KD-positive states need exactly this description; the tight entropic uncertainty relations of [10], for instance, are established under the hypothesis that the KD-positive set is the hull of the basis projectors [10, Lemma 2]. The answer depends on the basis pair. It is affirmative for almost every pair of bases [4], for the discrete Fourier transform in prime dimension [2] and in dimension p^2 [3], and for all cyclic groups of prime-power order [1]. It is negative for some spin-one bases [2, 5] and for the Fourier pairs of $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ [1]. De Bièvre, Langrenez and Radchenko state the case of an arbitrary finite abelian group as an open question [1, Section 8].

This note answers it: the cyclic groups of prime-power order are the only ones for which the answer is affirmative. The obstruction is a Bell-type inequality. Take $G = \mathbb{Z}_p \times \mathbb{Z}_q$ with p and q prime. The Hilbert space is $\mathbb{C}^p \otimes \mathbb{C}^q$, and each factor carries two bases, position and Fourier. A KD-positive state supplies a joint probability distribution for the four resulting outcomes, so it obeys every classical correlation bound, such as the Clauser–Horne–Shimony–Holt (CHSH) bound. Mixtures of pure KD-positive states obey a strictly stronger bound, because in each pure KD-positive state one of the two outcomes of each factor is uniformly distributed. Quantum mechanics offers states whose KD table is still a probability distribution but whose correlations violate the stronger bound; the room between the two bounds exists because Tsirelson's bound $2\sqrt{2}$ exceeds the classical value 2. In the simplest case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ the two bases of each qubit are the eigenbases of the Pauli matrices Z and X, and the state

$$\rho = \frac{1}{4}\Big(I - \frac{1}{3}\big(Z \otimes Z + X \otimes Z + X \otimes X - Z \otimes X\big)\Big) \tag{1}$$

is KD-positive without being a mixture of pure KD-positive states (Proposition 8). The general construction replaces Z and X by zero-mean observables that are diagonal in the two bases of $\mathbb{Z}_p$ and of $\mathbb{Z}_q$, and what it measures is the gap between the two correlation bounds.

Setting. Let G be a finite abelian group of cardinality N, with counting measure. Its position basis is $(|g\rangle)_{g\in G}$ and its Fourier basis is

$$|\chi\rangle = N^{-1/2} \sum_{g\in G} \chi(g)|g\rangle, \qquad \chi \in \widehat{G}.$$

For any operator A define the KD array

$$Q_G(A)(g,\chi) = \langle\chi|g\rangle\langle g|A|\chi\rangle = \frac{1}{N}\sum_{h\in G} A_{g,h}\chi(h-g). \tag{2}$$

The inverse and the Hilbert–Schmidt overlap identity are

$$A_{g,h} = \sum_{\chi\in\widehat{G}} Q_G(A)(g,\chi)\chi(g-h), \tag{3}$$

$$\operatorname{Tr}(A^*B) = N\sum_{g,\chi} \overline{Q_G(A)(g,\chi)}\,Q_G(B)(g,\chi). \tag{4}$$

Write $\mathcal{E}_G$ for the density operators with nonnegative real KD array and $\mathcal{C}_G = \operatorname{conv}\{P \in \mathcal{E}_G : P \text{ is a rank-one projector}\}$. We use the full trace norm $\|A\|_1 = \operatorname{Tr}\sqrt{A^*A}$, without a factor $1/2$.

The classification of pure KD-positive states due to De Bièvre, Langrenez and Radchenko [1, Theorem 3.1] is an input: their vectors, up to global phase, are

$$\psi(g) = |H|^{-1/2}\gamma(g)\mathbf{1}_{a+H}(g), \qquad H \le G, \quad a \in G, \quad \gamma \in \widehat{G}. \tag{5}$$

With $H^\perp = \{\chi \in \widehat{G} : \chi|_H = 1\}$, the projector P onto ψ has the array

$$Q_G(P) = \frac{1}{N}\,\mathbf{1}_{(a+H)\times(\gamma H^\perp)}, \tag{6}$$

supported on a "rectangle" with $|H|\,|H^\perp| = N$ points. By (4), for a Hermitian operator W with real array,

$$\mathrm{Tr}(WP) = \text{the average of } N\,Q_G(W) \text{ over the rectangle } (a+H)\times(\gamma H^\perp). \tag{7}$$

Results. The main statement is the following classification. The trivial group counts as cyclic of prime-power order.

Theorem 1. *Let G be a finite abelian group. Then $\mathcal{E}_G = \mathcal{C}_G$ if and only if G is cyclic of prime-power order.*

The "if" direction is [1, Theorem 6.1]. The "only if" direction follows from three results proved below. Theorem 2 shows that the zero extension from a subgroup and the pull-back from a quotient preserve the trace-norm distance to the pure-state hull; a finite abelian group that is not cyclic of prime-power order contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$ with p, q prime, so only these groups need a counterexample. Theorem 4 provides one for every pair of primes, by the CHSH mechanism described above. For $\mathbb{Z}_3 \times \mathbb{Z}_3$ the counterexample is the fully explicit state of Theorem 9, whose separation from the hull is certified in rational arithmetic with the distance bound 111/20000.

Versions. The first version of this note (17 November 2026) contained the exact $\mathbb{Z}_3 \times \mathbb{Z}_3$ certificate, the transfer theorem and the resulting classification for groups of order $2^a 3^b$. The present version adds Theorem 4 and, with it, the classification for all finite abelian groups. The earlier material is kept unchanged in Sections 3 and 5.

2 Related work

The first systematic study of mixed KD-positive states is due to Langrenez, Arvidsson-Shukur and De Bièvre. They proved that the KD-positive states are exactly the mixtures of the basis projectors for the discrete Fourier transform in prime dimension [2, Theorem 1.1(iii)], and they exhibited a pair of spin-one bases with KD-positive states outside the hull of the pure ones [2, Section 5.2]. For a generic pair of bases the picture is the simplest possible: with probability one the only pure KD-positive states are the basis vectors, and every KD-positive state is a mixture of them [4, Theorem 1]. Mixed KD-positive states outside the pure-state hull therefore require special basis pairs. For the spin-one pair, the convex roof of the total KD nonpositivity equals one exactly on the pure-state hull, so that it witnesses the separation faithfully [5, Section 4 and Appendix B].

The Fourier pairs are the natural structured case. For the discrete Fourier transform in dimension d, Yang, Yao, Geng, Wang and Chen proved the equality $\mathcal{E} = \mathcal{C}$ for $d = p^2$ [3, Theorem 1] and conjectured it for every d [3, Conjecture 1]. For $d = pq$ with distinct primes they showed that the operators with a real KD array are the real span of the pure KD-positive projectors [3, Lemma 4], and wrote that they believe the equality but could not prove it. De Bièvre, Langrenez and Radchenko moved to arbitrary finite abelian groups: they classified the pure KD-positive states [1, Theorem 3.1], reduced the equality $\mathcal{E}_G = \mathcal{C}_G$ to a decomposition property of positive periodic functions [1, Theorem 5.2], and proved the equality for every cyclic group of prime-power order [1, Theorem 6.1]. They refuted the conjecture of [3] by separated states

for $\mathbb{Z}_6$ and for $\mathbb{Z}_2 \times \mathbb{Z}_2$, obtained by perturbing a state on a facet of the hull along its outward normal, with positivity checked numerically [1, Section 7], and left the general abelian case open [1, Section 8]. Cai, Yang and Zheng gave a second proof of the prime-power case based on a directed graph, proved the equality on the real span of the pure states attached to any path of that graph, and stated the structure for general d as an open question [6, Theorems 1 and 2, Conclusion]. Burkat and Strelchuk, classifying the unitaries that preserve KD positivity, record that for $d = pq \neq 6$ the equality remained an open question to the best of their knowledge [7, Appendix D]. Thio et al. found exotic states, KD-positive and outside the pure-state hull, for the multiqubit groups $\mathbb{Z}_2^n$ with $n \geq 2$ by a related facet search [9]. Spriet extended the pure-state classification to second-countable locally compact abelian groups and conjectured that the equality holds when the compact open subgroups are totally ordered by inclusion, a condition that "might also be necessary" [8, discussion after Eq. (12)]. For finite groups this condition singles out the cyclic groups of prime-power order.

Against this background, Theorem 1 settles the finite case completely. It refutes [3, Conjecture 1] for every $d = pq$ with distinct primes, answers the open question of [1, Section 8], and proves that Spriet's condition is necessary for finite groups. The new cases are the groups $\mathbb{Z}_p \times \mathbb{Z}_q$ with $pq \notin \{4, 6\}$ and everything that contains them; the subgroup reduction itself is elementary. The witness of Section 4 is of a different nature from the facet perturbations used so far: it is given in closed form for all primes at once and identifies the obstruction as a Bell inequality. The first version of this note treated $\mathbb{Z}_3 \times \mathbb{Z}_3$ by the facet construction of [1, Section 7] with an exact certificate; that example is kept because it comes with an explicit distance to the hull.

A search of the literature citing [1] and of public repositories of machine-generated proofs, made on 23 November 2026, found no earlier proof of Theorem 4 or of Theorem 1. This statement concerns bibliographic priority and is independent of the mathematical verification described in Section 7.

3 Transfer between a group, its subgroups and its quotients

The following specialization of a channel-retraction argument carries a counterexample from a subgroup or a quotient to the whole group, with its distance to the hull. Its general norm-contraction mechanism is standard; the point to verify is the action on the KD hulls.

Theorem 2. *Let $H \leq G$ and $K = G/H$, with $m = |H|$. Let $V : \ell^2(H) \to \ell^2(G)$ be zero extension and let $J : \ell^2(K) \to \ell^2(G)$ be the isometry $(Jf)(g) = m^{-1/2} f(g + H)$. Each isometry sends KD-positive states to KD-positive states and sends its pure-state hull into $\mathcal{C}_G$. Moreover, for $E = V$ with $T = H$, or $E = J$ with $T = K$, and every density operator ω on $\ell^2(T)$,*

$$\operatorname{dist}_1(E\omega E^*, \mathcal{C}_G) = \operatorname{dist}_1(\omega, \mathcal{C}_T). \tag{8}$$

Proof. Equation (2) first gives the two formulas

$$Q_G(V\omega V^*)(g,\chi) = \begin{cases} \dfrac{m}{N} Q_H(\omega)(g, \chi|_H), & g \in H, \\ 0, & g \notin H, \end{cases} \tag{9}$$

$$Q_G(J\omega J^*)(g,\chi) = \begin{cases} \dfrac{1}{m} Q_K(\omega)(g + H, \bar{\chi}), & \chi|_H = 1, \\ 0, & \chi|_H \neq 1. \end{cases} \tag{10}$$

Here $\bar{\chi}$ is the induced character of K. These prove preservation of KD positivity. Formula (5) proves preservation of pure KD-positive projectors: for V, extend a character of H to G; for J, take the preimage in G of the subgroup and coset in K.

We next check compression of a pure KD-positive vector ψ of G. For $V^*\psi$, its support is the intersection of a subgroup coset $a + L$ with H. This is empty or a coset of $L \cap H$, so the compression is zero or proportional to a vector in (5) for H. For $J^*\psi$, sum the character on each intersection of a fibre of $G \to K$ with $a + L$. Such an intersection is a coset of $H \cap L$. If the character is nontrivial on $H \cap L$, the sum vanishes. Otherwise its magnitude is constant on the coset $\pi(a + L)$, where $\pi : G \to K$ is the quotient map, and its phase is a character of $\pi(L)$, extendible to K. This compression is again zero or proportional to a pure KD-positive vector of K.

Fix any pure $\tau \in \mathcal{C}_T$. The completely positive trace-preserving map

$$\mathcal{R}_E(X) = E^*XE + \mathrm{Tr}[(I - EE^*)X]\tau$$

satisfies $\mathcal{R}_E(\mathcal{C}_G) \subseteq \mathcal{C}_T$ by the compression check, and $\mathcal{R}_E(E\omega E^*) = \omega$. The inclusion $E\mathcal{C}_T E^* \subseteq \mathcal{C}_G$ and trace-norm invariance under an isometry give the "$\leq$" direction of (8). For the reverse direction, every $\sigma \in \mathcal{C}_G$ satisfies

$$\|\omega - \mathcal{R}_E(\sigma)\|_1 = \|\mathcal{R}_E(E\omega E^* - \sigma)\|_1 \leq \|E\omega E^* - \sigma\|_1,$$

by channel contraction on Hermitian operators. Taking infima finishes the proof. □

Corollary 3. *Let T be a subgroup or a quotient of G, or a quotient of a subgroup. If $\mathcal{E}_T \neq \mathcal{C}_T$, then $\mathcal{E}_G \neq \mathcal{C}_G$. Moreover, a finite abelian group that is not cyclic of prime-power order contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$ for some primes $p \leq q$.*

Proof. The hull $\mathcal{C}_T$ is a polytope, hence closed, so a state $\omega \in \mathcal{E}_T \setminus \mathcal{C}_T$ lies at positive distance from it, and Theorem 2 gives a state of $\mathcal{E}_G$ at the same positive distance from $\mathcal{C}_G$. For the second assertion, if two distinct primes $p < q$ divide $|G|$, Cauchy's theorem provides elements of orders p and q, which generate a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$. Otherwise G is a non-cyclic p-group, a direct sum of at least two nontrivial cyclic groups, and elements of order p in two summands generate $\mathbb{Z}_p \times \mathbb{Z}_p$. □

The sets $\mathcal{E}_G$ and $\mathcal{C}_G$ are defined by the group structure alone: a group isomorphism induces a unitary that maps position basis to position basis and Fourier basis to Fourier basis. The group $\mathbb{Z}_p \times \mathbb{Z}_q$ with $p \neq q$ may therefore be replaced by the cyclic group $\mathbb{Z}_{pq}$ whenever convenient.

4 A CHSH witness for $\mathbb{Z}_p \times \mathbb{Z}_q$

Theorem 4. *For all primes $p \leq q$, the group $G = \mathbb{Z}_p \times \mathbb{Z}_q$ satisfies $\mathcal{E}_G \neq \mathcal{C}_G$.*

The case $p = q = 3$ is Theorem 9 of Section 5. This section proves all the others.

4.1 The witness and the candidate state

Here $N = pq$. Write a point of $G \times \widehat{G}$ as $x = (x_1, x_2, x_3, x_4)$, where $(x_1, x_2) \in \mathbb{Z}_p \times \mathbb{Z}_q$ is the position and $(x_3, x_4) \in \mathbb{Z}_p \times \mathbb{Z}_q$ labels the character $\chi_{(x_3,x_4)}(g_1, g_2) = e^{2\pi i(x_3 g_1/p + x_4 g_2/q)}$. The Fourier basis of G is the tensor product of the Fourier bases $(|f_\alpha\rangle)$ of $\mathbb{C}^p$ and $(|f_\beta\rangle)$ of $\mathbb{C}^q$. Let

$u_1, u_3 : \mathbb{Z}_p \to [-1,1]$ and $u_2, u_4 : \mathbb{Z}_q \to [-1,1]$ be functions with zero mean, and set

$$A_0 = \sum_{a\in\mathbb{Z}_p} u_1(a)|a\rangle\langle a|, \qquad A_1 = \sum_{\alpha\in\mathbb{Z}_p} u_3(\alpha)|f_\alpha\rangle\langle f_\alpha|,$$
$$B_0 = \sum_{b\in\mathbb{Z}_q} u_2(b)|b\rangle\langle b|, \qquad B_1 = \sum_{\beta\in\mathbb{Z}_q} u_4(\beta)|f_\beta\rangle\langle f_\beta|,$$
$$\mathcal{B} = A_0\otimes B_0 + A_1\otimes B_0 + A_1\otimes B_1 - A_0\otimes B_1, \qquad \mathcal{W} = I + \mathcal{B}, \qquad \rho_c = \frac{I - c\,\mathcal{B}}{N} \quad (c>0). \tag{11}$$

The observables A_0, A_1 act on the first factor and B_0, B_1 on the second; $\mathcal{B}$ is a CHSH operator, $\mathcal{W}$ is the witness and ρ_c the candidate state. Two families of functions are used. The *alternating* function is $v_2 = (1,-1)$ for the prime 2, and $v_p(0) = 0$, $v_p(x) = (-1)^x$ for $1 \le x \le p-1$ when p is odd. If $p \neq q$ we take $u_1 = u_3 = v_p$ and $u_2 = u_4 = v_q$. If $p = q$ is odd we take the Legendre symbol $u_1 = u_2 = u_3 = \left(\frac{\cdot}{p}\right)$ and $u_4 = -\left(\frac{-1}{p}\right)\left(\frac{\cdot}{p}\right)$. For a function u on $\mathbb{Z}_p$ put $\|u\|^2 = \frac{1}{p}\sum_x u(x)^2$; this equals 1 for v_2 and $(p-1)/p$ for v_p and for the Legendre symbol with p odd. Let

$$S(x) = u_1(x_1)u_2(x_2) + u_3(x_3)u_2(x_2) + u_3(x_3)u_4(x_4) - u_1(x_1)u_4(x_4),$$
$$\nu = (\|u_1\|^2 + \|u_3\|^2)(\|u_2\|^2 + \|u_4\|^2). \tag{12}$$

Lemma 5. *With the notation above:*

(a) $N\,Q_G(\mathcal{W}) = 1 + S$ *and* $N^2 Q_G(\rho_c) = 1 - c\,S$, *with* $|S| \le 2$. *Hence* $Q_G(\rho_c) \ge 0$ *whenever* $c \le \frac{1}{2}$.

(b) If $(p,q) \neq (2,2)$, *then* $\mathrm{Tr}(\mathcal{W}P) \ge 0$ *for every pure KD-positive projector* P *of* G.

(c) $\mathrm{Tr}(\mathcal{W}\rho_c) = 1 - c\,\nu$.

Proof. (a) In $\langle g|(A_i\otimes B_j)|\chi\rangle$, let an observable that is diagonal in the position basis act on $\langle g|$ and one that is diagonal in the Fourier basis act on $|\chi\rangle$. This gives $Q_G(A_0\otimes B_0)(x) = u_1(x_1)u_2(x_2)/N$, $Q_G(A_1\otimes B_0)(x) = u_3(x_3)u_2(x_2)/N$, and similarly for the two other terms, while $Q_G(I) = 1/N$. Finally $|S| = |u_2(u_1+u_3) + u_4(u_3-u_1)| \le |u_1+u_3| + |u_3-u_1| \le 2$.

(b) By (7), $\mathrm{Tr}(\mathcal{W}P)$ is the average of $1+S$ over a rectangle $(a+H)\times(\gamma H^\perp)$. For $H = 0$, the position (x_1,x_2) is fixed and (x_3,x_4) is uniform on $\mathbb{Z}_p\times\mathbb{Z}_q$; the zero means remove three of the four terms of S and leave $1 + u_1(x_1)u_2(x_2) \ge 0$. In the same way $H = G$, $H = \mathbb{Z}_p\times 0$ and $H = 0\times\mathbb{Z}_q$ give $1+u_3u_4$, $1+u_3u_2$ and $1-u_1u_4$, all nonnegative. For $p \neq q$ there are no other subgroups. For $p = q$ odd, the remaining subgroups are the lines $L_s = \{(t, st)\}$ with $s \neq 0$, for which $L_s^\perp = \{(x_3,x_4) : x_3 = -sx_4\}$. On a rectangle attached to L_s one has $x_2 = sx_1 + d$ with x_1 uniform and $x_3 = -sx_4 + e$ with x_4 uniform and independent of x_1. The two mixed terms of S average to zero and

$$\mathrm{Tr}(\mathcal{W}P) = 1 + \mathbb{E}\big[u_1(x_1)u_2(sx_1+d)\big] + \mathbb{E}\big[u_3(-sx_4+e)u_4(x_4)\big].$$

For $k \neq 0$, the identity $\sum_x \left(\frac{x(x+k)}{p}\right) = -1$ holds, because for $x \neq 0$ one has $\left(\frac{x(x+k)}{p}\right) = \left(\frac{1+k/x}{p}\right)$ and $1 + k/x$ runs over $\mathbb{F}_p \setminus \{1\}$. Therefore $\frac{1}{p}\sum_x \left(\frac{x}{p}\right)\left(\frac{sx+d}{p}\right)$ equals $\left(\frac{s}{p}\right)\frac{p-1}{p}$ if $d = 0$ and $-\left(\frac{s}{p}\right)\frac{1}{p}$ if $d \neq 0$. The first expectation is thus $\left(\frac{s}{p}\right)\frac{p-1}{p}$ or $-\left(\frac{s}{p}\right)\frac{1}{p}$. By the choice of u_4 and $\left(\frac{-1}{p}\right)\left(\frac{-s}{p}\right) = \left(\frac{s}{p}\right)$, the second expectation is $-\left(\frac{s}{p}\right)\frac{p-1}{p}$ or $\left(\frac{s}{p}\right)\frac{1}{p}$. Whatever the sign of $\left(\frac{s}{p}\right)$, one of the two expectations is at least $-\frac{1}{p}$ and the other at least $-\frac{p-1}{p}$, so $\mathrm{Tr}(\mathcal{W}P) \ge 0$.

(c) The zero means give $\mathrm{Tr}\,A_i = \mathrm{Tr}\,B_j = 0$ and $\mathrm{Tr}(A_0A_1) = \sum_{a,\alpha} u_1(a)u_3(\alpha)|\langle a|f_\alpha\rangle|^2 = \frac{1}{p}\sum_a u_1(a)\sum_\alpha u_3(\alpha) = 0$, and likewise $\mathrm{Tr}(B_0B_1) = 0$. Hence $\mathrm{Tr}\,\mathcal{B} = 0$, the cross terms of $\mathrm{Tr}\,\mathcal{B}^2$ vanish, and $\frac{1}{N}\mathrm{Tr}\,\mathcal{B}^2 = \nu$. Then $\mathrm{Tr}(\mathcal{W}\rho_c) = \frac{1}{N}\mathrm{Tr}((I+\mathcal{B})(I-c\mathcal{B})) = 1 - c\nu$. □

The operator ρ_c is positive semidefinite when $c\,\lambda_{\max}(\mathcal{B}) \le 1$, where $\lambda_{\max}$ denotes the largest eigenvalue. Two bounds on $\lambda_{\max}(\mathcal{B})$ are used.

Lemma 6. *Let A_0, A_1 and B_0, B_1 be Hermitian contractions on two finite-dimensional Hilbert spaces and let $\mathcal{B}$ be as in* (11)*. Then $\lambda_{\max}(\mathcal{B}) \le 2\sqrt{2}$. Moreover $\lambda_{\max}(\mathcal{B}) \le 2\kappa_A$, where $\kappa_A = \max_\theta \|\cos\theta\, A_0 + i \sin\theta\, A_1\|$ in operator norm, and symmetrically $\lambda_{\max}(\mathcal{B}) \le 2\kappa_B$ with $\kappa_B = \max_\theta \|\cos\theta\, B_0 + i\sin\theta\, B_1\|$.*

Proof. Write B_j for $I \otimes B_j$. For a unit vector ψ, the Cauchy–Schwarz inequality gives

$$|\langle\psi, \mathcal{B}\psi\rangle| \le \|(B_0 - B_1)\psi\| + \|(B_0 + B_1)\psi\| \le \sqrt{2}\,\langle\psi, (2B_0^2 + 2B_1^2)\psi\rangle^{1/2} \le 2\sqrt{2},$$

which is Tsirelson's bound. For the second bound, dilate each B_j to the Hermitian unitary

$$R_j = \begin{pmatrix} B_j & \sqrt{I - B_j^2} \\ \sqrt{I - B_j^2} & -B_j \end{pmatrix}$$

on the doubled space. Both R_0 and R_1 compress to B_0 and B_1 under the same isometry, so $\lambda_{\max}(\mathcal{B}) \le \lambda_{\max}(\mathcal{B}')$, where $\mathcal{B}'$ is built from A_0, A_1, R_0, R_1. Two Hermitian unitaries R_0, R_1 decompose their space into common invariant subspaces of dimension one or two: if w is an eigenvector of the unitary R_0R_1 with eigenvalue λ, then $R_1 w = \lambda R_0 w$ and $R_1R_0 w = \bar\lambda w$, so the span of w and $R_0 w$ is invariant under both, and so is its orthogonal complement. On a one-dimensional block $R_0 = r_0$ and $R_1 = r_1$ are signs and $\mathcal{B}' = (r_0 - r_1)A_0 + (r_0 + r_1)A_1$ is $\pm 2A_0$ or $\pm 2A_1$, of norm at most $2\kappa_A$ (take $\theta = 0$ or $\pi/2$). A two-dimensional block on which R_0 or R_1 is $\pm I$ splits into one-dimensional blocks. On the others, $R_0 = \sigma_z$ and $R_1 = \cos t\,\sigma_z + \sin t\,\sigma_x$ in a suitable basis. Then $R_0 - R_1 = 2\sin\frac{t}{2}\,\sigma_a$ and $R_0 + R_1 = 2\cos\frac{t}{2}\,\sigma_b$ for two anticommuting Pauli matrices σ_a, σ_b, and in a basis where $\sigma_a = \sigma_x$ and $\sigma_b = \pm\sigma_y$,

$$\mathcal{B}' = 2\begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}, \qquad M = \sin\tfrac{t}{2}\,A_0 \pm i\cos\tfrac{t}{2}\,A_1,$$

whose largest eigenvalue is $2\|M\| \le 2\kappa_A$. Exchanging the roles of the two factors gives the bound with κ_B. □

Proposition 7. *Let $(p, q) \ne (2, 2)$, let $\Lambda \ge \lambda_{\max}(\mathcal{B})$ and $c = \min(\frac{1}{2}, \frac{1}{\Lambda})$. If $\nu > \max(2, \Lambda)$, then $\rho_c \in \mathcal{E}_G \setminus \mathcal{C}_G$.*

Proof. The operator ρ_c has trace one and is positive semidefinite since $c\Lambda \le 1$; it is KD-positive by Lemma 5(a). By Lemma 5(c), $\mathrm{Tr}(\mathcal{W}\rho_c) = 1 - c\nu < 0$, whereas $\mathrm{Tr}(\mathcal{W}\sigma) \ge 0$ for every $\sigma \in \mathcal{C}_G$ by Lemma 5(b). □

4.2 Case analysis

Put $s_2 = 2$ and $s_p = 2(p-1)/p$ for odd p, so that $\nu = s_p s_q$ for the functions chosen above; s_p increases with p over the odd primes. Let κ_3 and κ_5 be the constants κ_A of Lemma 6 for the alternating functions v_3 and v_5. The computation described in Section 7 certifies $\kappa_3 \le 1.25642$ and $\kappa_5 \le 1.28772$.

pairs (p, q)	lower bound on ν	Λ	bound used
$(2, 3)$	$8/3 \approx 2.667$	$2\kappa_3 \le 2.5129$	κ bound on the $\mathbb{Z}_3$ factor
$(2, q)$, $q \ge 5$	$16/5 = 3.2$	$2\sqrt{2} \approx 2.8284$	Tsirelson
$(3, q)$, $q \ge 19$	$48/19 \approx 2.5263$	$2\kappa_3 \le 2.5129$	κ bound on the $\mathbb{Z}_3$ factor
$(5, q)$, $q \ge 7$	$96/35 \approx 2.7429$	$2\kappa_5 \le 2.5755$	κ bound on the $\mathbb{Z}_5$ factor
(p, q), $7 \le p < q$	$240/77 \approx 3.1169$	$2\sqrt{2}$	Tsirelson
(p, p), $p \ge 7$	$144/49 \approx 2.9388$	$2\sqrt{2}$	Tsirelson

In every row $\nu > \max(2, \Lambda)$, so Proposition 7 applies and proves Theorem 4 for these pairs. The pairs left over are $(2,2)$, $(3,3)$, $(5,5)$, $(3,5)$, $(3,7)$, $(3,11)$, $(3,13)$ and $(3,17)$.

4.3 The group $\mathbb{Z}_2 \times \mathbb{Z}_2$

For two qubits the witness $\mathcal{W}$ takes the value -1 on one Bell state, and a term $Y \otimes Y$ repairs it.

Proposition 8. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, so that $A_0 = B_0 = Z$ and $A_1 = B_1 = X$, and let $\mathcal{W}' = I + \mathcal{B} - Y \otimes Y$. Then $\mathrm{Tr}(\mathcal{W}'P) \geq 0$ on the twenty pure KD-positive projectors, and for $\frac{1}{4} < c \leq \frac{\sqrt{2}}{4}$ the operator ρ_c is a KD-positive state with $\mathrm{Tr}(\mathcal{W}'\rho_c) = 1 - 4c < 0$. The state (1) is $\rho_{1/3}$.*

Proof. The subgroups 0, $\mathbb{Z}_2 \times 0$, $0 \times \mathbb{Z}_2$ and G give sixteen product states. On each of them exactly one of the four correlators of $\mathcal{B}$ equals ± 1, the three others vanish, and $\langle Y \otimes Y \rangle = 0$, so $\mathrm{Tr}(\mathcal{W}'P) \in \{0, 2\}$. The diagonal subgroup $\{00, 11\}$ gives the four Bell states, on which $(\langle Z \otimes Z \rangle, \langle X \otimes X \rangle, \langle Y \otimes Y \rangle)$ is $(1,1,-1)$, $(1,-1,1)$, $(-1,1,1)$ or $(-1,-1,-1)$ and the two mixed correlators vanish; the values of $\mathrm{Tr}(\mathcal{W}'P)$ are $4, 0, 0, 0$. The spectrum of $\mathcal{B}$ is $\{0, 0, \pm 2\sqrt{2}\}$, so $\rho_c \geq 0$ for $c \leq \sqrt{2}/4$, and $c \leq \frac{1}{2}$ gives KD positivity by Lemma 5(a). Finally $\mathrm{Tr}(\mathcal{B}\rho_c) = -c\nu = -4c$ and $\mathrm{Tr}((Y \otimes Y)\rho_c) = 0$. □

4.4 Six pairs settled by explicit states

For $(3,5)$, $(3,7)$, $(3,11)$, $(3,13)$, $(3,17)$ and $(5,5)$ the witness $\mathcal{W}$ is kept and ρ_c is replaced by a state of the form

$$\rho = \sum_{a,b} w_{12}(a,b)\, |a\rangle\langle a| \otimes |b\rangle\langle b| + \sum_{\alpha,b} w_{32}(\alpha,b)\, |f_\alpha\rangle\langle f_\alpha| \otimes |b\rangle\langle b| + \sum_{\alpha,\beta} w_{34}(\alpha,\beta)\, |f_\alpha\rangle\langle f_\alpha| \otimes |f_\beta\rangle\langle f_\beta| + \sum_{a,\beta} w_{14}(a,\beta)\, |a\rangle\langle a| \otimes |f_\beta\rangle\langle f_\beta|,$$

a real combination of the product pure KD-positive projectors, with rational weights of denominator 10^7 that sum to one and may be negative. By (6),

$$N\, Q_G(\rho)(x) = w_{12}(x_1, x_2) + w_{32}(x_3, x_2) + w_{34}(x_3, x_4) + w_{14}(x_1, x_4),$$

and $\mathrm{Tr}(\mathcal{W}\rho)$ is the corresponding weighted sum of the values $1 \pm u_i u_j$ of Lemma 5(b). Both are exact rational numbers. The weights were obtained by semidefinite programming, mixed with the maximally mixed state to gain a positivity margin, and rounded; the proof uses only the stored weights. The checker verifies in exact arithmetic that $Q_G(\rho) \geq 0$, that $\mathrm{Tr}(\mathcal{W}\rho) < 0$, and again that $\mathrm{Tr}(\mathcal{W}P) \geq 0$ on every pure KD-positive projector, and certifies $\rho > 0$ in interval (ball) arithmetic.

(p,q)	$\mathrm{Tr}(\mathcal{W}\rho)$	$\min N\, Q_G(\rho)$	certified lower bound on $\lambda_{\min}(\rho)$
$(3,5)$	$-246111/2500000$	$333/2500000$	1.33×10^{-4}
$(3,7)$	$-663653/5000000$	$951/10000000$	9.52×10^{-5}
$(3,11)$	$-334063/2000000$	$151/2500000$	6.05×10^{-5}
$(3,13)$	$-35569/200000$	$4/78125$	5.12×10^{-5}
$(3,17)$	$-943767/5000000$	$391/10000000$	3.91×10^{-5}
$(5,5)$	$-2181631/10000000$	$799/10000000$	7.99×10^{-5}

Together with Proposition 7, the case table, Proposition 8 and Theorem 9 below, this proves Theorem 4.

5 An exact certificate for $\mathbb{Z}_3 \times \mathbb{Z}_3$

For $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ the conjugate basis is obtained by applying the three-dimensional Fourier transform to each qutrit, namely $F_3 \otimes F_3$; this is different from the Fourier transform F_9 of the cyclic group $\mathbb{Z}_9$, for which $\mathcal{E} = \mathcal{C}$. Each of the six subgroups of G gives nine pure KD-positive states, for a total of 54, which include both product and entangled states. The following statement gives a separated state by the facet construction of [1, Section 7], with a certificate that uses rational arithmetic alone and yields a distance to the hull. In this section W denotes the witness of the theorem, normalized so that the hull lies below the value 18.

Theorem 9. *For the position/Fourier pair of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, there exist an explicitly specified density operator $\rho > 0$ and a Hermitian operator W such that*

$$\min Q_G(\rho) = \frac{2291}{1935000} > 0, \tag{13}$$

$$\max_{\sigma \in \mathcal{C}_G} \mathrm{Tr}(W\sigma) = 18, \qquad \mathrm{Tr}(W\rho) = \frac{206973}{10000} \approx 20.6973, \tag{14}$$

$$\mathrm{dist}_1(\rho, \mathcal{C}_G) \geq \frac{111}{20000}. \tag{15}$$

In particular, $\mathcal{C}_G \subsetneq \mathcal{E}_G$.

Identify G with $\mathbb{F}_3^2$ and put $\zeta = e^{2\pi i/3}$. Rows and columns are both ordered lexicographically as $(0,0), (0,1), (0,2), (1,0), \ldots, (2,2)$; column c represents the character $\chi_c(g) = \zeta^{c \cdot g}$. Define ρ and W by (3) from

$$Q_G(\rho) = \frac{R}{7740000}, \qquad Q_G(W) = M, \tag{16}$$

where

$$R = \begin{pmatrix}
99613 & 123096 & 99613 & 99613 & 123096 & 99613 & 123096 & 56130 & 103096 \\
99613 & 99613 & 123096 & 99613 & 99613 & 123096 & 123096 & 103096 & 56130 \\
123096 & 123096 & 123096 & 123096 & 123096 & 123096 & 96130 & 76130 & 76130 \\
99613 & 99613 & 123096 & 123096 & 76130 & 123096 & 123096 & 79613 & 79613 \\
99613 & 99613 & 123096 & 123096 & 99613 & 99613 & 123096 & 56130 & 103096 \\
103096 & 56130 & 79613 & 56130 & 79613 & 103096 & 76130 & 9164 & 32647 \\
123096 & 99613 & 99613 & 99613 & 123096 & 99613 & 123096 & 103096 & 56130 \\
123096 & 123096 & 76130 & 99613 & 123096 & 99613 & 123096 & 79613 & 79613 \\
56130 & 103096 & 79613 & 103096 & 79613 & 56130 & 76130 & 32647 & 9164
\end{pmatrix}, \tag{17}$$

and

$$M = \begin{pmatrix}
-1 & 8 & -1 & -1 & 8 & -1 & 8 & -10 & 8 \\
-1 & -1 & 8 & -1 & -1 & 8 & 8 & 8 & -10 \\
8 & 8 & 8 & 8 & 8 & 8 & -10 & -10 & -10 \\
-1 & -1 & 8 & 8 & -10 & 8 & 8 & -1 & -1 \\
-1 & -1 & 8 & 8 & -1 & -1 & 8 & -10 & 8 \\
8 & -10 & -1 & -10 & -1 & 8 & -10 & -28 & -19 \\
8 & -1 & -1 & -1 & 8 & -1 & 8 & 8 & -10 \\
8 & 8 & -10 & -1 & 8 & -1 & 8 & -1 & -1 \\
-10 & 8 & -1 & 8 & -1 & -10 & -10 & -19 & -28
\end{pmatrix}. \tag{18}$$

5.1 Hermiticity, normalization and positivity

All entries of R are positive and their sum is 7740000. Equation (3) yields $\rho = A + B\zeta$ with rational matrices A, B. Direct substitution, using $\zeta^2 = -1 - \zeta$, gives

$$A_{ji} = A_{ij} - B_{ij}, \qquad B_{ji} = -B_{ij},$$

so ρ is Hermitian. The same check applies to W. Also $\operatorname{Tr}\rho = 1$, and the smallest entry of R is 9164, proving (13).

Here is a short rational certificate for $\rho > 0$. For $i \neq j$ set

$$u_{ij} = |A_{ij} - B_{ij}/2| + \frac{7}{8}|B_{ij}|.$$

Since $\rho_{ij} = (A_{ij} - B_{ij}/2) + i\sqrt{3}B_{ij}/2$ and $\sqrt{3}/2 < 7/8$, one has $|\rho_{ij}| \leq u_{ij}$. For $s = (1,1,1,1,1,2,1,1,2)$, exact rational arithmetic gives

$$\begin{aligned} m_i &= s_i\rho_{ii} - \sum_{j\neq i} s_j u_{ij}, \\ (m_i)_{i=1}^9 &= \frac{1}{61920000}(1360135, 1360135, 1376216, 940135, 940135, \\ &\qquad\qquad 330636, 786094, 786094, 536024). \end{aligned} \tag{19}$$

Every entry is strictly positive. With $D = \operatorname{diag}(s)$, the Hermitian matrix $D\rho D$ has positive diagonal and is strictly diagonally dominant, hence is positive definite. Equivalently, Gershgorin applied to $D^{-1}\rho D$ gives the useful explicit bound

$$\lambda_{\min}(\rho) \geq \min_i \frac{m_i}{s_i} = \frac{27553}{10320000} > 0. \tag{20}$$

The much larger numerical eigenvalue estimate is not used in this proof.

5.2 Separation from all pure KD-positive mixtures

The subgroups of $\mathbb{F}_3^2$ are the zero subgroup, four lines, and the full group. For each subgroup H the distinct pure projectors in (5) have the arrays (6), with $N = 9$. There are $|G/H||\widehat{G}/H^\perp| = 9$ of them for each H, and hence 54 in total. Each rectangle has nine occupied entries. By (4), $\operatorname{Tr}(WP)$ is the sum of the entries of M on that rectangle. Enumerating the six subgroups and their cosets gives the multiset

$$\{\operatorname{Tr}(WP) : P \text{ pure KD-positive}\} = \{18 \text{ (43 times)},\ -63 \text{ (10 times)},\ -144 \text{ (1 time)}\}. \tag{21}$$

This is a finite integer calculation, not a numerical optimization. Linearity proves that the maximum over $\mathcal{C}_G$ is 18. On the other hand, direct substitution gives

$$\operatorname{Tr}(W\rho) = \frac{9}{7740000}\sum_{g,c} R_{g,c}M_{g,c} = \frac{206973}{10000} \approx 20.6973 > 18.$$

This proves (14) and excludes ρ from $\mathcal{C}_G$.

For completeness the trace-norm bound has a conservative rational certificate as well. Write $W = A^{(W)} + B^{(W)}\zeta$ by (3). Then

$$\|W - 18I\|_\infty \leq \max_i \sum_j (|A_{ij}^{(W)} - 18\delta_{ij}| + |B_{ij}^{(W)}|) = 486.$$

Here Hermiticity allows the maximum absolute row sum to bound the operator norm. For any $\sigma \in \mathcal{C}_G$, the trace of $\rho - \sigma$ vanishes and Hölder's inequality yields

$$486\|\rho - \sigma\|_1 \geq \operatorname{Tr}[(W - 18I)(\rho - \sigma)] \geq \frac{26973}{10000}.$$

Since $26973/(486 \cdot 10000) = 111/20000$, this completes the proof of Theorem 9.

Remark 10. *The arrays also admit a geometric construction. If ρ_0 is the uniform average of the 43 pure projectors with score 18 in (21), then $Q_G(\rho) = Q_G(\rho_0) + M/20000$. This is the facet-perturbation construction of [1, Section 7.2]; the witness was found numerically, and the positivity proof above is independent of that search.*

6 Proof of Theorem 1 and consequences

Proof of Theorem 1. If G is cyclic of prime-power order, $\mathcal{E}_G = \mathcal{C}_G$ by [1, Theorem 6.1]. Otherwise G contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$ with $p \le q$ prime (Corollary 3), for which the equality fails by Theorem 4, and the failure passes to G by Corollary 3. □

Theorem 2 also transports distances, which gives a quantitative statement wherever an explicit distance is available.

Corollary 11. *Every finite abelian group with a $\mathbb{Z}_3^2$ subquotient admits a state in $\mathcal{E}_G$ at trace-norm distance at least 111/20000 from $\mathcal{C}_G$. In particular this holds for all noncyclic finite abelian 3-groups.*

Proof. Compose the subgroup and quotient isometries in Theorem 2. A noncyclic abelian 3-group contains a subgroup $\mathbb{Z}_3^2$. □

Remark 12. *For groups of order $2^a 3^b$ the classification already follows from Theorem 2, Theorem 9 and the two examples of [1, Section 7]: a noncyclic group has a noncyclic Sylow subgroup, which contains $\mathbb{Z}_2^2$ or $\mathbb{Z}_3^2$, and a cyclic group with $a, b > 0$ contains $\mathbb{Z}_6$. This was the content of the first version of this note; it relied on the numerical positivity checks of [1] for $\mathbb{Z}_6$ and $\mathbb{Z}_2^2$, which Proposition 8 and the first row of the case table now replace by proofs.*

Remark 13. *The state ρ_c has a joint probability distribution for the four product bases, so it satisfies the classical CHSH bound $\mathrm{Tr}(\mathcal{B}\rho_c) \ge -2$, while mixtures of pure KD-positive states satisfy $\mathrm{Tr}(\mathcal{B}\sigma) \ge -1$ for $(p, q) \ne (2, 2)$. In the decomposition criterion of [1, Theorem 5.2], the obstruction comes from the four-cycle formed by the subgroups 0, $\mathbb{Z}_p \times 0$, G and $0 \times \mathbb{Z}_q$, which is absent when the subgroups form a chain, as they do for $\mathbb{Z}_{p^k}$. The analogous question for infinite locally compact abelian groups, including the sufficiency conjecture of [8], is not addressed here.*

7 Verification materials

The proofs of Theorems 2 and 9, Lemmas 5 and 6 and Propositions 7 and 8 are written out above. Three steps of Theorem 4 are computer-assisted: the bounds on κ_3 and κ_5, the six states of Section 4.4, and, as a convenience, the arithmetic of Theorem 9. The public bundle contains the corresponding checkers and data.

For $\mathbb{Z}_3 \times \mathbb{Z}_3$, standard-library Python programs reconstruct all 54 pure arrays, verify strict positivity and the witness value, and check the diagonal-dominance certificate in exact arithmetic: the weighted diagonal-dominance checker, the independent cyclotomic checker, and the manuscript-data checker, together with the exact certificate.

For the CHSH witness, the state checker reads the stored rational weights of the six states (files `chsh-state-p-q.json`) and verifies, in exact rational arithmetic, the trace, the nonnegativity of the KD array, the sign of $\mathrm{Tr}(\mathcal{W}\rho)$ and the nonnegativity of $\mathcal{W}$ on every pure KD-positive projector. Positive definiteness is certified with Arb ball arithmetic at 160 bits: the matrix is conjugated by a floating-point approximation U of its eigenvectors, and $U^*\rho U$ is shown to have a positive diagonal and to be strictly diagonally dominant, which implies $\rho > 0$ whatever the

quality of the approximation. The κ checker uses $\|\cos\theta\, A_0 + i\sin\theta\, A_1\|^2 = \lambda_{\max}(C_0 + \cos 2\theta\, C_1 + \sin 2\theta\, C_2)$ with $C_0 = \frac{1}{2}(A_0^2 + A_1^2)$, $C_1 = \frac{1}{2}(A_0^2 - A_1^2)$ and $C_2 = \frac{i}{2}[A_0, A_1]$; it certifies an upper bound at each of 4000 equally spaced angles by the same ball-arithmetic test and adds the Lipschitz term $\frac{\pi}{4000}(\|C_1\|_F + \|C_2\|_F)$, where $\|\cdot\|_F$ is the Frobenius norm. The analytic cross-checker verifies Proposition 8 symbolically, the rational inequalities of the case table, and Lemma 5(b) exhaustively on thirteen pairs up to $(13, 13)$. The common routines are in `chsh_common.py`. These files complement the proofs given above.

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