Local discrimination of maximally entangled two-qubit states: the separable optimum and when one message attains it

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Abstract

Two distant parties share one of k equally likely maximally entangled states of two qubits and must guess which one. Each party alone holds a completely mixed qubit and learns nothing, so every bit of information has to come from correlating the two local measurements. We determine exactly how well this can be done. Maximally entangled two-qubit states correspond to lines in $\mathbb{R}^4$, and the answer depends on a single number, the angular radius Θ of the smallest cone of lines that contains the family. The optimal success probability over all separable measurements is $(1+\sin 2\Theta)/k$ when $\Theta \le \pi/4$ and $2/k$ otherwise; for two qubits this is also the optimum over measurements with positive partial transpose. A separate exact formula gives the best protocol in which one party measures and sends a single classical bit. One message attains the separable optimum for every family of at most three states, where the result is therefore the exact optimum of local operations and classical communication. It fails for four states at the vertices of a tetrahedron, for instance with values 0.4182 against 0.4268. For the five equiangular states of Wei, Cobucci and Tavakoli, one bit reaches exactly 2/5, above the value of about 0.395 that their numerical search had found without communication and had not improved with one-way protocols.

1 Introduction

Local state discrimination asks how much of the information stored in a shared quantum state can be read by parties who measure their own subsystems and exchange classical messages. The restriction is severe for entangled states. The four Bell states are orthogonal, so a joint measurement identifies them with certainty, yet no protocol of local operations and classical communication (LOCC) can do so [1], and the best one guesses the right Bell state with probability 1/2 [9, 15]. The reason is visible in the reduced states: each party holds the completely mixed qubit state whichever Bell state was prepared.

This note treats the general version of that example for two qubits. Alice and Bob share one of k maximally entangled states $|e_1\rangle, \dots, |e_k\rangle$, drawn with equal probabilities. The states need not be orthogonal, and k is arbitrary. We ask for the largest probability of guessing the label in three settings of increasing power: one party measures and sends its outcome to the other (one-way LOCC), the parties run an arbitrary LOCC protocol, and the measurement is only required to be separable, meaning that every measurement operator is a sum of positive product operators. Separable measurements contain all LOCC measurements and admit a much simpler mathematical characterization [2].

Both extreme settings have exact answers. A maximally entangled two-qubit state is fixed, up to a sign, by a unit vector $q \in \mathbb{R}^4$, and the overlap of two such states is the Euclidean

inner product of their vectors. Let Θ be the smallest angle such that some line in $\mathbb{R}^4$ makes an angle at most Θ with all k lines $\pm q_j$. Theorem 6 shows that the separable optimum is

$$P_{\mathrm{SEP}} = \frac{1 + \sin\bigl(2\min(\Theta, \pi/4)\bigr)}{k}.$$

Two states make this concrete: Θ is half the angle between the two lines, and the formula is the Helstrom bound $\frac{1}{2}(1 + \sqrt{1 - |\langle e_1|e_2\rangle|^2})$. The general statement says that k states are as hard to separate locally as their tightest common cone is narrow, and that nothing is gained once the cone is wider than $\pi/4$, where the trace bound $2/k$ is reached. Theorem 4 gives the one-way optimum as $P_{\rightarrow} = (1 + \sqrt{1 - \mu^2})/k$, where μ is the smallest singular value found in the convex hull of the rotation matrices attached to the states. Theorem 7 compares the two. One message is optimal exactly when the states that touch the boundary of the tightest cone can be balanced within a plane, which always happens for $k \leq 3$. For those families $P_{\rightarrow} = P_{\mathrm{LOCC}} = P_{\mathrm{SEP}}$, and the LOCC optimum is known in closed form. Four states at the vertices of a regular tetrahedron violate the balance condition, and one message is then strictly weaker than a separable measurement.

Section 2 places these statements in the literature. Section 3 fixes the parametrisation, Sections 4 to 6 contain the three theorems, and Section 7 works out three families, including the five-state measurement of Wei, Cobucci and Tavakoli [3].

2 Related work

The optimality conditions of minimum-error discrimination form a semidefinite duality, stated for instance as Theorem 1 of Deconinck and Terhal [4]; it is the tool of Section 4. For qubit states this duality has a geometric solution on the Bloch sphere [4, 5], which for equally likely pure states is the smallest ball enclosing the Bloch vectors; Lemma 3 restates this case with a short proof.

Local discrimination of entangled states started with the result of Walgate, Short, Hardy and Vedral that two orthogonal pure states can always be distinguished perfectly by LOCC [6]. Virmani, Sacchi, Plenio and Markham extended it to the optimal discrimination of two arbitrary pure states and noted that two maximally entangled states can be discriminated locally as well as globally [7]; this is the case $k = 2$ of our theorems. For orthogonal maximally entangled states, perfect local distinguishability was studied by Ghosh et al. [1, 8] and bounds on the LOCC error by Nathanson [9]. Nathanson's Lemma 8 already bounds the LOCC success probability for k equally likely states $(\mathbf{1} \otimes U_j)|\psi\rangle$ of $\mathbb{C}^D \otimes \mathbb{C}^D$ by D/k, without assuming orthogonality, and Bandyopadhyay and Nathanson proved the underlying inequality $\langle\psi|T|\psi\rangle \leq \lambda_1 \operatorname{Tr} T$, with λ_1 the largest Schmidt coefficient of $|\psi\rangle$, for every operator $0 \leq T \leq \mathbf{1}$ with positive partial transpose [12, Lemma 2]. For two qubits this is the trace bound $2/k$ that our separable formula reaches when $\Theta \geq \pi/4$, and in dimension D it is the upper half of Proposition 10. Bandyopadhyay, Ghosh and Kar related perfect one-way LOCC discrimination of such states to the perfect discrimination of the unitaries that generate them [10], and Nathanson exhibited triples in higher dimensions that one-way LOCC cannot distinguish perfectly although two-way LOCC can [11]. The gap between one-way and separable measurements found here for four non-orthogonal two-qubit states is of the same nature; whether two-way LOCC closes it is open.

Separable and positive-partial-transpose (PPT) measurements are the standard relaxations of LOCC [2]. Semidefinite programming gives exact values for specific sets, for instance the PPT value $7/8$ for the four states of Yu, Duan and Ying [13, 14], whose separable value is $3/4$ [15], and the separable optimum for three or four Bell states assisted by a partially entangled

pair [15]; without assistance the latter reduces to 2/3 and 1/2, which are the orthogonal cases of Theorem 6. Chitambar, Duan and Hsieh proved that almost all ensembles of three pure two-qubit states cannot be discriminated by LOCC as well as globally [16]. Corollary 8 supplies the exact LOCC value for the triples made of maximally entangled states.

The closest antecedent of the application in Section 7.3 is the work of Wei, Cobucci and Tavakoli [3], who construct five equiangular maximally entangled two-qubit states and report numerical local discrimination values; Proposition 10 shows that one message reaches the known upper bound for their set. We did not find the formulas of Theorems 4–7 in the works above or in searches on local discrimination of non-orthogonal maximally entangled states and on the equivalent problem of discriminating qubit unitaries with an unentangled probe, for which the classical results concern two unitaries [17, 18]. This search supports the positioning of the note and does not certify priority.

3 Setting

3.1 States, rotations and vectors in $\mathbb{R}^4$

Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ be the Pauli matrices and write $\sigma_a = a \cdot \sigma$ for $a \in \mathbb{R}^3$. For a unit vector $q = (q_0, \vec{q}) \in \mathbb{R}^4$, regarded as a unit quaternion, put

$$U(q) = q_0 \mathbf{1} - i\, \vec{q} \cdot \sigma \in SU(2), \qquad U(q)\, \sigma_a\, U(q)^\dagger = \sigma_{R(q)a},$$

which defines a rotation $R(q) \in SO(3)$. The map $q \mapsto U(q)$ turns the quaternion product into the matrix product, so $R(pq) = R(p)R(q)$ and $R(\bar{q}) = R(q)^{\mathsf{T}}$ for the conjugate $\bar{q} = (q_0, -\vec{q})$. If $q = (\cos\varphi, \sin\varphi\, n)$ with a unit vector n, then $R(q)$ is the rotation by the angle 2φ about n:

$$R(q)a = \cos 2\varphi\; a + \sin 2\varphi\; n \times a + (1 - \cos 2\varphi)(n \cdot a)\, n. \tag{1}$$

With the singlet $|S\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, every maximally entangled state of two qubits is, up to a phase, of the form

$$|e(q)\rangle = (\mathbf{1} \otimes U(q))|S\rangle.$$

The vectors q and $-q$ give the same state up to a sign, so the state is a line in $\mathbb{R}^4$. Since $\langle S|A \otimes \mathbf{1}|S\rangle = \mathrm{Tr}(A)/2$, the overlap is the Euclidean inner product, $\langle e(p)|e(q)\rangle = \mathrm{Tr}(U(p)^\dagger U(q))/2 = p \cdot q$. The singlet has $\langle S|\sigma_a \otimes \sigma_b|S\rangle = -a \cdot b$ and completely mixed marginals, hence

$$\langle e(q)|\sigma_a \otimes \sigma_b|e(q)\rangle = -\, b \cdot R(q)a, \qquad \langle e(q)|\sigma_a \otimes \mathbf{1}|e(q)\rangle = \langle e(q)|\mathbf{1} \otimes \sigma_b|e(q)\rangle = 0. \tag{2}$$

When Alice projects her qubit on the $+1$ eigenstate of σ_a, Bob is therefore left with the pure state of Bloch vector $-R(q)a$.

Throughout, $q_1, \ldots, q_k$ are unit vectors of $\mathbb{R}^4$, $|e_j\rangle = |e(q_j)\rangle$ and $R_j = R(q_j)$. The states are equally likely. A measurement is a family $(M_1, \ldots, M_k)$ of positive operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ summing to the identity, and its success probability is $\frac{1}{k}\sum_j \langle e_j|M_j|e_j\rangle$. We write $P_{\to}$, P_{LOCC}, P_{SEP} and P_{PPT} for the optimum over one-way LOCC measurements with Alice measuring first, all LOCC measurements, separable measurements, and measurements whose operators have a positive partial transpose. These classes are nested, so $P_{\to} \le P_{\mathrm{LOCC}} \le P_{\mathrm{SEP}} \le P_{\mathrm{PPT}}$. A positive operator on two qubits with positive partial transpose is separable [19], hence $P_{\mathrm{SEP}} = P_{\mathrm{PPT}}$ in this note.

3.2 The covering angle

Definition 1. The covering angle $\Theta \in [0, \pi/2]$ of the family is defined by

$$\cos\Theta = \max_{\kappa\in\mathbb{R}^4,\ \|\kappa\|=1}\ \min_{1\le j\le k} |\kappa \cdot q_j|.$$

A maximiser κ is called a centre.

Thus Θ is the smallest angular radius of a cone of lines around some axis κ that contains all the lines $\pm q_j$. Given a centre κ, write the relative quaternions as

$$\bar{\kappa}\, q_j = \varepsilon_j\,(\cos\varphi_j,\ \sin\varphi_j\, n_j), \qquad \varepsilon_j = \pm 1, \quad 0 \le \varphi_j \le \Theta,$$

with unit vectors $n_j \in \mathbb{R}^3$, so that $\cos\varphi_j = |\kappa \cdot q_j|$ and $W_j := R(\kappa)^{\mathsf{T}} R_j$ is the rotation by $2\varphi_j$ about n_j. A state is active if $\varphi_j = \Theta$.

Lemma 2 (Balance)**.** *Let $0 < \Theta < \pi/2$ and let κ be a centre. There are weights $w_j \ge 0$ on the active states, with $\sum_j w_j = 1$, such that $\sum_j w_j n_j = 0$.*

Proof. Otherwise the origin lies outside the convex hull of the active n_j and there is a unit vector τ with $\tau \cdot n_j > 0$ for every active j. Put $\kappa_\epsilon = \kappa\,(\cos\epsilon, \sin\epsilon\,\tau)$. Then $|\kappa_\epsilon \cdot q_j| = \cos\epsilon\cos\varphi_j + \sin\epsilon\sin\varphi_j\,\tau\cdot n_j$ for small $\epsilon > 0$. This exceeds $\cos\Theta$ to first order in ϵ for the active states, because $\sin\Theta > 0$, and stays above $\cos\Theta$ for the others, which contradicts the maximality of κ. □

4 One classical message

The one-way problem reduces to a qubit discrimination problem on Bob's side, for which we recall the geometric solution.

Lemma 3 (Equally likely pure qubit states)**.** *Let $b_1, \dots, b_k$ be unit vectors of $\mathbb{R}^3$, let z be the point of $\operatorname{conv}\{b_j\}$ closest to the origin and $\rho = \sqrt{1 - \|z\|^2}$. The optimal probability of identifying the equally likely qubit states $\frac{1}{2}(\mathbf{1} + \sigma_{b_j})$ is $(1+\rho)/k$. Moreover ρ is the radius of the smallest ball containing the b_j.*

Proof. The projection property of z gives $b_j \cdot z \ge \|z\|^2$, hence $\|b_j - z\|^2 \le 1 - \|z\|^2 = \rho^2$. The operator $Y = ((1+\rho)\mathbf{1} + \sigma_z)/(2k)$ therefore dominates every $(\mathbf{1} + \sigma_{b_j})/(2k)$, and any measurement (N_j) has success $\sum_j \operatorname{Tr}(N_j(\mathbf{1}+\sigma_{b_j}))/(2k) \le \operatorname{Tr} Y = (1+\rho)/k$. For the converse write $z = \sum_j w_j b_j$ with weights w_j. Every j with $w_j > 0$ satisfies $b_j \cdot z = \|z\|^2$, that is $\|b_j - z\| = \rho$. If $\rho > 0$, the operators $N_j = w_j(\mathbf{1} + \sigma_{(b_j - z)/\rho})$ are positive, sum to $\mathbf{1}$, and succeed with probability $\frac{1}{k}\sum_j w_j(1 + b_j \cdot (b_j - z)/\rho) = (1+\rho)/k$. If $\rho = 0$ all states coincide. Finally, for any centre c, $\sum_j w_j \|b_j - c\|^2 = \rho^2 + \|c - z\|^2$, so no ball of radius smaller than ρ contains the b_j. □

Theorem 4 (One-way optimum)**.** *Let Δ_k be the simplex of probability weights and*

$$\mu = \min_{w\in\Delta_k}\ \sigma_{\min}\Big(\sum_{j=1}^{k} w_j R_j\Big),$$

where $\sigma_{\min}$ denotes the smallest singular value. Then

$$P_{\to} = \frac{1 + \sqrt{1 - \mu^2}}{k}.$$

The optimum is attained when Alice measures a spin component σ_v and sends the resulting bit. Exchanging the roles of Alice and Bob gives the same value.

Proof. Alice's measurement can be refined into rank-one operators $c_i(\mathbf{1}+\sigma_{v_i})/2$ with $\sum_i c_i = 2$. By (2) the outcome i has probability $c_i/2$ under every hypothesis, so the states stay equally likely, and Bob holds the pure state of Bloch vector $-R_j v_i$. By Lemma 3 his optimal success is $(1+\rho(v_i))/k$ with $\rho(v)^2 = 1 - \min_w \|\sum_j w_j R_j v\|^2$. The overall success is the average $\sum_i (c_i/2)(1+\rho(v_i))/k \le (1+\max_v \rho(v))/k$. Since $\rho(-v) = \rho(v)$, the projective measurement of σ_v for a maximising v attains this bound. Minimising $\|\sum_j w_j R_j v\|$ over unit vectors v gives the smallest singular value, hence $\max_v \rho(v)^2 = 1 - \mu^2$. If Bob measures first, the matrices R_j are replaced by their transposes, which leaves μ unchanged. □

Alice's outcome $\pm$ along v prepares on Bob's side the state that the unitary $U(q_j)$ produces from the pure probe of Bloch vector $\mp v$. The value $P_{\to}$ is therefore also the optimal probability of identifying one of the k unitaries $U(q_j)$ with a single use and an unentangled probe, the minimum-error counterpart of the correspondence studied for perfect discrimination in [10].

5 Separable measurements

Lemma 5 (A bound by an operator-norm radius)**.** *For every real 3×3 matrix Λ,*

$$P_{\mathrm{SEP}} \le \frac{1}{k}\Big(1 + \max_j \|R_j - \Lambda\|\Big),$$

where $\|\cdot\|$ is the operator norm.

Proof. Every positive product operator is a sum of rank-one product operators, so the operators of a separable measurement can be written $M_j = \sum_i c_{ji}\, \frac{1}{2}(\mathbf{1} + \sigma_{a_{ji}}) \otimes \frac{1}{2}(\mathbf{1} + \sigma_{b_{ji}})$ with $c_{ji} > 0$ and unit vectors a_{ji}, b_{ji}. Completeness requires $\sum_{j,i} c_{ji} = 4$ and, for the components along $\sigma_\alpha \otimes \sigma_\beta$, $\sum_{j,i} c_{ji}\, b_{ji} a_{ji}^{\mathsf{T}} = 0$. By (2), $\langle e_j | M_j | e_j \rangle = \frac{1}{4}\sum_i c_{ji}(1 - b_{ji} \cdot R_j a_{ji})$. The completeness relation allows us to subtract Λ at no cost:

$$\frac{1}{k}\sum_j \langle e_j | M_j | e_j \rangle = \frac{1}{k}\Big(1 - \frac{1}{4}\sum_{j,i} c_{ji}\, b_{ji} \cdot (R_j - \Lambda) a_{ji}\Big) \le \frac{1}{k}\Big(1 + \max_j \|R_j - \Lambda\|\Big). \qquad \square$$

Theorem 6 (Separable optimum)**.** *For every family of k equally likely maximally entangled two-qubit states with covering angle Θ,*

$$P_{\mathrm{SEP}} = P_{\mathrm{PPT}} = \begin{cases} \dfrac{1 + \sin 2\Theta}{k}, & \Theta \le \pi/4, \\[2ex] \dfrac{2}{k}, & \Theta \ge \pi/4. \end{cases}$$

Proof. Upper bounds. Taking $\Lambda = 0$ in Lemma 5 gives $P_{\mathrm{SEP}} \le 2/k$, the known trace bound [9, 12]. Let $\Theta \le \pi/4$, let κ be a centre, $O = R(\kappa)$ and $c = \cos 2\Theta \ge 0$. With $W_j = O^{\mathsf{T}} R_j$,

$$(R_j - cO)^{\mathsf{T}}(R_j - cO) = (1 + c^2)\mathbf{1} - c\,(W_j + W_j^{\mathsf{T}}) \le (1 - c^2)\mathbf{1},$$

because (1) gives $\frac{1}{2}(W_j + W_j^{\mathsf{T}}) \ge \cos 2\varphi_j\, \mathbf{1} \ge c\, \mathbf{1}$. Hence $\|R_j - cO\| \le \sin 2\Theta$ and Lemma 5 with $\Lambda = cO$ yields $P_{\mathrm{SEP}} \le (1 + \sin 2\Theta)/k$.

Attainment for $\Theta \le \pi/4$. If $\Theta = 0$ all states coincide and the value $1/k$ is trivial. Otherwise take the weights w_j of Lemma 2. For each active j choose a unit vector $a_j \perp n_j$ and put $a'_j = n_j \times a_j$. By (1), $W_j a_j = \cos 2\Theta\, a_j + \sin 2\Theta\, a'_j$ and $W_j a'_j = \cos 2\Theta\, a'_j - \sin 2\Theta\, a_j$. Define

$$M_j = w_j\Big[\tfrac{1}{2}(\mathbf{1} + \sigma_{a_j} \otimes \sigma_{b_j}) + \tfrac{1}{2}(\mathbf{1} + \sigma_{a'_j} \otimes \sigma_{b'_j})\Big], \qquad b_j = -O a'_j, \quad b'_j = O a_j,$$

and $M_j = 0$ for the other states. Each bracket is separable, since $\frac{1}{2}(\mathbf{1} + \sigma_a \otimes \sigma_b)$ is the sum of the product projectors on the $(+,+)$ and $(-,-)$ eigenstates of (σ_a, σ_b). The coefficient matrix of the terms $\sigma_\alpha \otimes \sigma_\beta$ in $\sum_j M_j$ is $\frac{1}{2}\sum_j w_j(a_j b_j^{\mathsf{T}} + a'_j b'^{\mathsf{T}}_j) = \frac{1}{2}\sum_j w_j(a'_j a_j^{\mathsf{T}} - a_j a'^{\mathsf{T}}_j)O^{\mathsf{T}} = \frac{1}{2}[\sum_j w_j n_j]_\times O^{\mathsf{T}} = 0$, where $[n]_\times a = n \times a$. Thus $\sum_j M_j = \mathbf{1}$. By (2), $-b_j \cdot R_j a_j = a'_j \cdot W_j a_j = \sin 2\Theta$ and $-b'_j \cdot R_j a'_j = -a_j \cdot W_j a'_j = \sin 2\Theta$, so $\langle e_j|M_j|e_j\rangle = w_j(1 + \sin 2\Theta)$ and the success probability is $(1 + \sin 2\Theta)/k$.

Attainment for $\Theta \geq \pi/4$. We claim that there are positive semidefinite real matrices $S_1, \ldots, S_k$ with $\sum_j \operatorname{Tr} S_j = 1$ and $\sum_j R_j S_j = 0$. The set $\mathcal{C}$ of matrices $\sum_j R_j S_j$ with such S_j is compact and convex. If $0 \notin \mathcal{C}$, a separating hyperplane provides a matrix K and $\epsilon > 0$ with $\operatorname{Tr}(K^{\mathsf{T}} X) \geq \epsilon$ on $\mathcal{C}$; choosing $S_j = vv^{\mathsf{T}}$ shows that $v \cdot K^{\mathsf{T}} R_j v \geq \epsilon$ for all unit vectors v and all j. Then K is invertible, with polar decomposition $K = \tilde{O}P$, $\tilde{O}$ orthogonal and P positive definite, and $K^{\mathsf{T}} R_j = P\tilde{W}_j$ with $\tilde{W}_j = \tilde{O}^{\mathsf{T}} R_j$ orthogonal. For an eigenvector x of $\tilde{W}_j$ with eigenvalue λ,

$$0 < x^\dagger (P\tilde{W}_j + \tilde{W}_j^{\mathsf{T}} P)x = 2\,\mathrm{Re}(\lambda)\, x^\dagger P x,$$

so every eigenvalue of $\tilde{W}_j$ has a positive real part. A 3×3 orthogonal matrix with this property is a rotation by an angle smaller than $\pi/2$. Hence $\tilde{O} = R(\tilde{\kappa})$ for a unit quaternion $\tilde{\kappa}$, the rotation $\tilde{W}_j = R(\tilde{\kappa} q_j)$ has angle $2\tilde{\varphi}_j < \pi/2$ with $\cos\tilde{\varphi}_j = |\tilde{\kappa}\cdot q_j|$, and $|\tilde{\kappa}\cdot q_j| > \cos(\pi/4)$ for every j. This means $\Theta < \pi/4$, a contradiction. Given such matrices, diagonalise $S_j = \sum_i s_{ji} a_{ji} a_{ji}^{\mathsf{T}}$ and set $M_j = \sum_i s_{ji}(\mathbf{1} + \sigma_{a_{ji}} \otimes \sigma_{b_{ji}})$ with $b_{ji} = -R_j a_{ji}$. The identity components sum to $\mathbf{1}$, the correlation components sum to $-(\sum_j R_j S_j)$ contracted with $\sigma \otimes \sigma$, which vanishes, and $\langle e_j|M_j|e_j\rangle = 2\operatorname{Tr} S_j$. The success probability is $2/k$.

The equality $P_{\mathrm{SEP}} = P_{\mathrm{PPT}}$ was noted in Section 3. □

Lemma 5 and the two constructions show that $kP_{\mathrm{SEP}} - 1$ is the smallest radius of an operator-norm ball containing the rotations R_j. For $\Theta \leq \pi/4$ an optimal centre is $\cos 2\Theta\, R(\kappa)$, a contracted copy of the rotation attached to the centre of the cone.

6 When one message suffices

Theorem 7 (Comparison)**.** *The quantity μ of Theorem 4 satisfies $\mu \geq \max(0, \cos 2\Theta)$, and $P_{\to} = P_{\mathrm{SEP}}$ if and only if equality holds. For $0 < \Theta \leq \pi/4$ the following are equivalent.*

1. *$P_{\to} = P_{\mathrm{SEP}}$.*
2. *For some centre κ there are weights $w_j \geq 0$ on active states, summing to one, and a unit vector v such that $\sum_j w_j n_j = 0$ and $v \cdot n_j = 0$ whenever $w_j > 0$.*

In that case Alice measures σ_v, sends the sign $m = \pm 1$, and Bob applies the measurement $w_j(\mathbf{1} + m\,\sigma_{b_j})$ with $b_j = -R(\kappa)(n_j \times v)$.

Proof. The equivalence of $P_{\to} = P_{\mathrm{SEP}}$ with $\mu = \max(0, \cos 2\Theta)$ follows from the two formulas. Let $\Theta \leq \pi/4$, let κ be a centre and $O = R(\kappa)$. For a unit vector v and weights w, (1) gives

$$\Big\|\sum_j w_j R_j v\Big\| \;\geq\; v \cdot \sum_j w_j W_j v = \sum_j w_j\big[\cos 2\varphi_j + (1 - \cos 2\varphi_j)(n_j \cdot v)^2\big] \;\geq\; \cos 2\Theta, \qquad (3)$$

which proves $\mu \geq \cos 2\Theta$. If the second condition holds, then $\sum_j w_j W_j v = \cos 2\Theta\, v + \sin 2\Theta\,(\sum_j w_j n_j) \times v = \cos 2\Theta\, v$, so $\mu = \cos 2\Theta$. Conversely, let $\mu = \cos 2\Theta$ be attained at (w, v). Both inequalities in (3) are equalities. The second one forces $\varphi_j = \Theta$ and $n_j \cdot v = 0$ whenever $w_j > 0$. The first one, between a vector of norm $\cos 2\Theta$ and its component along

v, forces $\sum_j w_j W_j v = \cos 2\Theta\, v$, that is $\sin 2\Theta \left(\sum_j w_j n_j\right) \times v = 0$; as $\sum_j w_j n_j$ is orthogonal to v, it vanishes. The protocol is the specialisation of Lemma 3: the Bloch vectors $-R_j v$ of the weighted states have barycentre $z = -\cos 2\Theta\, Ov$ and $-R_j v - z = -\sin 2\Theta\, O(n_j \times v)$. □

Corollary 8 (At most three states)**.** *For $k \le 3$,*

$$P_{\to} = P_{\mathrm{LOCC}} = P_{\mathrm{SEP}} = \frac{1 + \sin\bigl(2\min(\Theta, \pi/4)\bigr)}{k}.$$

Proof. Let first $0 < \Theta \le \pi/4$ and take the weights of Lemma 2. Their support contains two or three active states. Two unit vectors with a vanishing positive combination are opposite, and three are linearly dependent; in both cases a unit vector v is orthogonal to all of them, and Theorem 7 applies. The case $\Theta = 0$ is trivial. Let now $\Theta > \pi/4$; this requires $k = 3$, since two lines have $\Theta \le \pi/4$. Choose a unit quaternion p orthogonal to q_1, q_2, q_3. Replacing q_j by $\bar{p} q_j$ multiplies every R_j on the left by a fixed rotation and changes neither μ nor Θ, and makes the real parts vanish, $q_j = (0, m_j)$, so that $R_j = 2 m_j m_j^{\mathsf{T}} - \mathbf{1}$ by (1). Put $N(w) = \sum_j w_j m_j m_j^{\mathsf{T}}$, a positive matrix of unit trace, and $\ell = \min_w \lambda_{\max}(N(w))$. By the minimax theorem, $\ell = \max_X \min_j m_j \cdot X m_j$ over real density matrices X, and an optimal X is supported on the top eigenspace of $N(w)$ for an optimal w. If $\ell > 1/2$ this eigenspace is a line spanned by a unit vector x, so $\min_j (x \cdot m_j)^2 = \ell > 1/2$ and the quaternion $(0, x)$ would give $\Theta < \pi/4$. Hence $\ell \le 1/2$. On the segment from an optimal w to a vertex of the simplex, $\lambda_{\max}(N)$ increases continuously to 1 and takes the value $1/2$, where $\sum_j w_j R_j = 2N - \mathbf{1}$ is singular. Thus $\mu = 0$ and $P_{\to} = 2/k = P_{\mathrm{SEP}}$. □

For $k = 2$ the corollary recovers the local attainability of the Helstrom bound for two maximally entangled states [7]. For $k \ge 4$ the balance condition can fail, as the next section shows. When $\Theta > \pi/4$ the comparison reduces to the question whether the convex hull of the rotations R_j contains a singular matrix; it does in every family we tested (Section 8), and we do not know whether this is general.

7 Three families

7.1 Three states with equal overlaps

Let t be a unit vector, u_1, u_2, u_3 unit vectors at 120° in the plane orthogonal to t, and for $0 \le r \le 1$

$$q_j = (0,\ \alpha t + \beta u_j), \qquad \alpha = \sqrt{\frac{1+2r}{3}}, \quad \beta = \sqrt{\frac{2(1-r)}{3}},$$

so that $\langle e_i | e_j \rangle = r$ for $i \ne j$. The three states interpolate between three Bell states ($r = 0$) and a single state ($r = 1$). The centre is $\kappa = (0, t)$ by symmetry, with $\cos\Theta = \alpha$, and Corollary 8 gives

$$P_{\to} = P_{\mathrm{LOCC}} = P_{\mathrm{SEP}} = \begin{cases} \dfrac{2}{3}, & 0 \le r \le \dfrac{1}{4}, \\[2ex] \dfrac{1 + 2\alpha\beta}{3}, & \dfrac{1}{4} \le r \le 1. \end{cases}$$

The joint optimum is $\bigl(\sqrt{1+2r} + 2\sqrt{1-r}\bigr)^2/9$. Indeed, let A be the matrix with columns $|e_j\rangle$; the Gram matrix $A^\dagger A$ has eigenvalues $1+2r, 1-r, 1-r$, the square-root measurement succeeds with the squared diagonal entry c^2 of $(A^\dagger A)^{1/2}$, where $c = (\sqrt{1+2r} + 2\sqrt{1-r})/3$, and the operator $Y = c\,(AA^\dagger)^{1/2}/3$ dominates every $|e_j\rangle\langle e_j|/3$ and has trace c^2. At $r = 1/2$ the local and joint values are 0.648 and $8/9 \approx 0.889$. Local measurements keep the value $2/3$ of three orthogonal Bell states up to the overlap $r = 1/4$.

7.2 Four states on a tetrahedron

Let $t_1, \dots, t_4$ be the unit vectors pointing to the vertices of a regular tetrahedron, and for $0 < \theta \le \pi/4$ let $q_j = (\cos\theta, \sin\theta\, t_j)$.

Proposition 9. *The tetrahedral family has $\Theta = \theta$ and*

$$\frac{1+\sin 2\Theta_3}{4} \;\le\; P_{\to} \;<\; P_{\mathrm{SEP}} = \frac{1+\sin 2\theta}{4}, \qquad \cos^2\Theta_3 = \cos^2\theta + \tfrac{1}{9}\sin^2\theta.$$

Proof. The overlaps $q_i \cdot q_j = \cos^2\theta - \frac{1}{3}\sin^2\theta$ are positive. If a unit vector κ had $|\kappa \cdot q_j| \ge \cos\theta$ for all j with two different signs, two of the vectors q_j would make an angle of at least $\pi - 2\theta \ge \pi/2$. With equal signs, $\min_j \kappa \cdot q_j$ is at most the average $\kappa_0 \cos\theta$, because $\sum_j t_j = 0$. Hence $\Theta = \theta$ with the unique centre $\kappa = \pm(1,0,0,0)$, all four states are active and $n_j = t_j$. No two of the t_j are opposite and any three are linearly independent, so the only balanced weights are uniform and no vector is orthogonal to all four t_j. Theorem 7 gives $P_{\to} < P_{\mathrm{SEP}}$, and Theorem 6 gives P_{SEP}. For the lower bound, restrict the weights in μ to three states. The same argument applied to q_1, q_2, q_3, whose average has norm $(\cos^2\theta + \frac{1}{9}\sin^2\theta)^{1/2} \ge \cos\theta$, shows that their covering angle is Θ_3, and Corollary 8 gives $\mu \le \cos 2\Theta_3$. □

Numerical optimisation of Alice's axis returns the lower bound to machine precision for the four values of θ tested, so the best single message appears to ignore one of the four states. Table 1 lists the values. The separable measurement of Theorem 6 uses eight operators of the form $\frac{1}{2}(\mathbf{1} + \sigma_a \otimes \sigma_b)$, two for each state. We do not know whether a two-way LOCC protocol attains it.

θ	0.2	$\pi/8$	0.6	$\pi/4$
One message, $(1+\sin 2\Theta_3)/4$	0.3420	0.4182	0.4753	0.4985
Separable, $(1+\sin 2\theta)/4$	0.3474	0.4268	0.4830	0.5000

Table 1: Four maximally entangled states on a tetrahedron.

7.3 The five-state simplex and tight frames

Wei, Cobucci and Tavakoli construct five maximally entangled two-qubit states with $|\langle e_i|e_j\rangle|^2 = 1/16$, the elements of a five-outcome non-projective Bell-state measurement [3]. For the discrimination of these states with one copy, Section VIII of their preprint reports a numerical value of about 0.3946 for local measurements without communication, no increase found with one-way LOCC, and $P_{\mathrm{PPT}} = 2/5$; the joint optimum is $4/5$. The following statement settles the values with communication.

Proposition 10. *Let $|e_1\rangle, \dots, |e_k\rangle$ be maximally entangled states of $\mathbb{C}^D \otimes \mathbb{C}^D$ with $\sum_j |e_j\rangle\langle e_j| = (k/D^2)\,\mathbf{1}$. Then $P_{\to} = P_{\mathrm{LOCC}} = P_{\mathrm{SEP}} = D/k$, and Alice may measure in any orthonormal basis.*

Proof. Let Alice obtain the outcome $|a\rangle$ of an orthonormal basis. Bob's conditional state under the hypothesis j is the pure projector $\tau_{j,a} = D\,\langle a|(|e_j\rangle\langle e_j|)|a\rangle$, and $\sum_j \tau_{j,a} = (k/D)\,\mathbf{1}$. The operators $(D/k)\,\tau_{j,a}$ form a measurement whose success probability, conditional on a and j, is $(D/k)\,\mathrm{Tr}\,\tau_{j,a}^2 = D/k$. Conversely a product vector has overlap at most $1/\sqrt{D}$ with a maximally entangled state, so a separable operator M satisfies $\langle e_j|M|e_j\rangle \le \mathrm{Tr}(M)/D$, and summing over the measurement gives $P_{\mathrm{SEP}} \le D/k$. This upper bound is known [9, 12]; the content of the proposition is that one message attains it for tight frames. □

The five states attain the Welch bound for equiangular vectors in dimension four and therefore form a tight frame, $\sum_j |e_j\rangle\langle e_j| = \frac{5}{4}\mathbf{1}$, an identity that the checker also confirms directly. For the simplex

$$P_{\to} = P_{\mathrm{LOCC}} = P_{\mathrm{SEP}} = P_{\mathrm{PPT}} = \frac{2}{5}.$$

One bit from Alice, obtained in an arbitrary basis, reaches the PPT value, and the ratio to the joint optimum $4/5$ is exactly $1/2$, the same as for the Bell basis. The value without communication is left open by this argument. In the parametrisation of Section 3 the family is $q_0 = (1, 0, 0, 0)$, $q_j = (\frac{1}{4}, \frac{\sqrt{15}}{4} t_j)$, with covering angle $\Theta \approx 1.150 > \pi/4$, in agreement with Theorem 6.

8 Verification and scope

The theorems are proved above. The public Python checker tests their ingredients in floating-point arithmetic, and its recorded output is part of the same entry. For each family it computes Θ, assembles the separable measurement of Theorem 6 and the one-way measurement of Theorem 4 as 4×4 matrices, and checks positivity, positivity of the partial transposes, completeness and the success probabilities; for $\Theta \le \pi/4$ it also checks the operator-norm certificate of the upper bound. The families are those of Section 7, pairs of states, the Bell states, and sixty random families with $2 \le k \le 6$. Among the random families, those with $k \le 3$ have $P_{\to} = P_{\mathrm{SEP}}$ to rounding error, and three with $k \ge 4$ show a gap, the largest being 3.5×10^{-4}. A separate exploratory run on three hundred random families with $4 \le k \le 6$ found $P_{\to} = 2/k$ in all 158 families with $\Theta > \pi/4$ and a gap in 17 of the 142 others. During the preparation of the note the separable values were also compared with a semidefinite program over PPT measurements on ninety random families, with a largest deviation of 3×10^{-8}; this comparison is not part of the public bundle.

The results concern a single copy, equal prior probabilities, maximally entangled pure states of two qubits, and parties without shared ancillary entanglement. Three questions are left open: the LOCC optimum when one message is not enough, as in the tetrahedral family; whether $P_{\to} = 2/k$ whenever $\Theta > \pi/4$; and the extension of Theorem 6 beyond two qubits, where Proposition 10 covers only tight frames. The bibliographic comparison of Section 2 is a targeted search and does not establish priority.

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