A Mpemba mode invisible to every strict reduction

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Abstract

In the Mpemba effect a system that starts farther from equilibrium reaches it sooner than one that starts closer. In Markovian open quantum systems the effect is governed by the slowest relaxation mode of the generator: a state with a small component along that mode can overtake a state that starts closer to equilibrium but carries a larger slow component. Recent work proposes to witness the effect through the expectation value of a single observable. We ask how large the support of such an observable must be in a many-body system, and show that it can be forced to be the whole system. We give an explicit Lindbladian on a chain of $n \geq 2$ qubits, with jump operators acting on one site or on two neighbouring sites, whose stationary state is unique and whose slowest mode is the global parity $Z_1 \cdots Z_n$, at rate 2ϵ with $0 < \epsilon < 1$. The full spectrum is $\{-2(m + \epsilon b)\}$ with multiplicity $\binom{2n-1}{m}$, for $0 \leq m \leq 2n-1$ and $b \in \{0, 1\}$. The state $(\mathbf{1} + \alpha Z_1 \cdots Z_n)/2^n$ has maximally mixed reductions on every strict subset of sites at all times, yet it relaxes at the slowest rate, and it is overtaken by the state $(\mathbf{1} + \beta X_n)/2^n$ with $\beta > \alpha$ at time $\log(\beta/\alpha)/(2 - 2\epsilon)$, in trace distance, in relative entropy and in every unitarily invariant distance to equilibrium that grows with the deviation from the maximally mixed state. Any observable sensitive to the slow amplitude acts on all n sites.

1 Introduction

A cup of hot water left outside can freeze before a cup of warm water. This counterintuitive ordering, the Mpemba effect, has a clean formulation for Markovian dynamics [1]. Relaxation towards a unique stationary state is a superposition of decaying modes, and at late times only the slowest one matters. Two initial states are then ranked by the size of their component along the slowest mode, not by their initial distance to equilibrium. A state farther from equilibrium but with a smaller slow component eventually becomes the closer one, and the two distance curves cross. In open quantum systems, Carollo, Lasanta and Lesanovsky [2] used this mechanism to accelerate relaxation exponentially, by applying a unitary rotation that removes the component of the initial state along the slowest mode.

Detecting the effect requires access to the slow component. Choquehuanca, Obando, Sarandy and de Paula [3] proposed to witness the Mpemba crossing with the expectation value of a single Hermitian observable, whose expansion on the eigenmodes of the generator is tuned to the slow and fast modes, and illustrated it on a single qubit. In a many-body system the natural follow-up question concerns the support of such an observable. If the slowest mode were visible in the state of one site or of a few neighbouring sites, a local probe would suffice. This note asks whether a local and primitive dynamics on a chain can have a slow mode that is global in the strongest sense, that is, invisible in the reduced state of every strict subset of sites.

The answer is yes, and the construction is elementary. The dynamics on n qubits combines nearest-neighbour bit flips X_jX_{j+1}, single-site dephasing Z_j, and a weak bit flip X_1 at rate ϵ on the first site. Without the boundary term the global parity $P = Z_1 \cdots Z_n$ commutes with every jump operator and would be conserved; the weak boundary term breaks this symmetry and turns P into the unique slowest mode, at rate 2ϵ. Since P is a product of traceless factors on all sites, every partial trace over a nonempty set of sites annihilates it. The state $\rho_P = (\mathbf{1} + \alpha P)/2^n$ is therefore indistinguishable from equilibrium on every strict subsystem at all times, while it is the slowest-relaxing state of the chain. The state $\rho_Q = (\mathbf{1} + \beta X_n)/2^n$, with $\beta > \alpha$, starts farther from equilibrium, visibly so on the last site, relaxes at rate 2, and overtakes ρ_P at $t^* = \log(\beta/\alpha)/(2 - 2\epsilon)$.

The main results are the exact spectrum of the generator (Proposition 1), the uniqueness of the stationary state and of the slowest mode (Corollary 2), the invisibility of the slow mode in strict reductions (Proposition 3), and the Mpemba crossing with its time (Theorem 4). The example separates two notions that are easily conflated. The slow amplitude can be estimated by measuring every qubit locally in the Z basis and multiplying the outcomes, so local measurements suffice once their records are correlated; what fails is every observable, and every reduced state, whose support is a strict subset of the chain.

2 Related work

Lu and Raz [1] formulated the Mpemba effect and its inverse for Markovian dynamics and derived a sufficient condition for it within nonequilibrium thermodynamics. We work in the same Markovian setting, where late-time relaxation is controlled by the slowest eigenmode of the generator. Carollo, Lasanta and Lesanovsky [2] brought the effect to Markovian open quantum systems, where the generator has the Gorini–Kossakowski–Sudarshan–Lindblad form, and showed that a unitary rotation making the initial state orthogonal to the slowest mode yields an exponential speed-up, with applications to Dicke models and all-to-all interacting spins. Such a rotation is cheap in the present example: a Hadamard gate on site 1 maps P to the faster string $X_1Z_2 \cdots Z_n$. What is global here is the detection of the slow component, not its removal.

Choquehuanca, Obando, Sarandy and de Paula [3] witness the quantum Mpemba effect through a single observable whose eigenmode coefficients are tuned to the slow and fast modes, and they demonstrate the construction on a qubit under generalized amplitude damping; many-body extensions are left to future work. Their construction uses a nonzero coefficient on the slowest mode. The construction below shows that in a local many-body dynamics this coefficient can force the observable to act on every site (Remark 5), and it gives a fully explicit many-body test case in which the slow and fast modes and the crossing time are known in closed form.

3 The dynamics and its spectrum

For a Hermitian unitary U on $(\mathbb{C}^2)^{\otimes n}$ let $\mathcal{D}_U(\rho) = U\rho U^\dagger - \rho$, the dissipator of a single jump operator U at unit rate, since $U^\dagger U = \mathbf{1}$. For $n \geq 2$ and $0 < \epsilon < 1$ consider

$$\mathcal{L} = \sum_{j=1}^{n-1} \mathcal{D}_{X_jX_{j+1}} + \epsilon\, \mathcal{D}_{X_1} + \sum_{j=1}^{n} \mathcal{D}_{Z_j}, \tag{1}$$

where X_j, Y_j, Z_j are the Pauli matrices on site j. Every jump operator acts on one site or on two neighbouring sites, so $\mathcal{L}$ is local. A Pauli string is a product $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_n$ with

$\sigma_j \in \{I, X, Y, Z\}$. For such a string define its dephasing weight $k(\sigma) = \#\{j : \sigma_j \in \{X, Y\}\}$, its flip pattern $f_j(\sigma) = 1$ if $\sigma_j \in \{Y, Z\}$ and 0 otherwise, and its number of domain walls $e(\sigma) = \#\{j \leq n-1 : f_j(\sigma) \neq f_{j+1}(\sigma)\}$.

Proposition 1. *Every Pauli string is an eigenvector of $\mathcal{L}$:*

$$\mathcal{L}(\sigma) = -r(\sigma)\,\sigma, \qquad r(\sigma) = 2k(\sigma) + 2e(\sigma) + 2\epsilon f_1(\sigma).$$

Consequently $\mathcal{L}$ is diagonalizable with real spectrum consisting of the numbers $-2(m + \epsilon b)$, $0 \leq m \leq 2n-1$, $b \in \{0, 1\}$, the eigenvalue $-2(m + \epsilon b)$ having multiplicity $\binom{2n-1}{m}$.

Proof. Two Pauli strings either commute or anticommute. If U is a Pauli string, $U\sigma U^\dagger = \sigma$ when they commute and $-\sigma$ when they anticommute, so $\mathcal{D}_U(\sigma)$ equals 0 or -2σ. The jump Z_j anticommutes with σ exactly when $\sigma_j \in \{X, Y\}$, which contributes $2k(\sigma)$. The jump X_1 anticommutes with σ exactly when $f_1(\sigma) = 1$, which contributes $2\epsilon f_1(\sigma)$. The jump $X_j X_{j+1}$ anticommutes with σ exactly when exactly one of σ_j, σ_{j+1} anticommutes with X, that is, when $f_j(\sigma) \neq f_{j+1}(\sigma)$, which contributes $2e(\sigma)$. The 4^n Pauli strings form a basis of the operator space, so $\mathcal{L}$ is diagonal in that basis.

For the multiplicities, a Pauli string is determined by its flip pattern $f \in \{0, 1\}^n$ together with the choice, on each site, between the two letters with that flip value, one of which is dephased (X or Y) and one not (I or Z). The dephasing weight is therefore independent of the flip pattern, and k takes the value k on $\binom{n}{k}$ choices. A flip pattern is determined by f_1 and the positions of its domain walls among the $n-1$ bonds, so for each value b of f_1 there are $\binom{n-1}{e}$ patterns with e walls. The number of strings with $r = 2(m + \epsilon b)$ is thus $\sum_{k+e=m} \binom{n}{k}\binom{n-1}{e} = \binom{2n-1}{m}$ by Vandermonde's identity. Since $\epsilon \notin \mathbb{Z}$, distinct pairs (m, b) give distinct eigenvalues. □

Corollary 2. *The identity spans the kernel of $\mathcal{L}$, so $\mathbf{1}/2^n$ is the unique stationary state, and every other eigenvalue has strictly negative real part; the dynamics is primitive. The spectral gap is 2ϵ, the eigenvalue -2ϵ is simple with eigenvector $P = Z_1 Z_2 \cdots Z_n$, and every other nonzero eigenvalue is at most -2. In particular the slowest non-stationary mode is unique and global.*

Proof. By Proposition 1 the eigenvalues 0 and -2ϵ correspond to $(m, b) = (0, 0)$ and $(0, 1)$, each with multiplicity $\binom{2n-1}{0} = 1$; all others have $m \geq 1$ and are at most $-2 < -2\epsilon$. A string with $m = 0$ has $k = 0$ and $e = 0$, so it uses only I and Z with a constant flip pattern: it is $\mathbf{1}$ when $f_1 = 0$ and P when $f_1 = 1$. □

The role of the boundary term is now clear. For $\epsilon = 0$ the parity P commutes with every jump operator and the stationary space is spanned by $\mathbf{1}$ and P. The term $\epsilon\mathcal{D}_{X_1}$ breaks this conservation law weakly and leaves P as a long-lived mode; the condition $\epsilon < 1$ makes it slower than every other mode, the next rate being 2, attained for instance by X_n, which has $k = 1$, $e = 0$ and $f_1 = 0$.

4 Invisibility in strict reductions

For a subset $A \subsetneq \{1, \ldots, n\}$ write $\mathrm{Tr}_{\bar{A}}$ for the partial trace over the complementary sites $\bar{A} \neq \emptyset$, and expand an operator as $\rho = 2^{-n} \sum_\sigma c_\sigma(\rho)\,\sigma$ with $c_\sigma(\rho) = \mathrm{Tr}(\sigma\rho)$. By Corollary 2, the amplitude of the slowest mode in ρ is $c_P(\rho) = \mathrm{Tr}(P\rho)$, and under the dynamics $c_P(e^{t\mathcal{L}}\rho) = e^{-2\epsilon t} c_P(\rho)$.

Proposition 3. *For every strict subset A of sites, $\mathrm{Tr}_{\bar{A}}\, P = 0$. Consequently:*

1. *for every state ρ and every real δ such that $\rho + \delta P/2^n \geq 0$, the states ρ and $\rho + \delta P/2^n$ have the same reductions to every strict subset of sites but slow amplitudes differing by δ;*

2. *for $\rho_P = (\mathbf{1} + \alpha P)/2^n$ with $|\alpha| \leq 1$, the evolved state $e^{t\mathcal{L}}\rho_P = (\mathbf{1} + \alpha e^{-2\epsilon t} P)/2^n$ has maximally mixed reductions $\mathbf{1}_A/2^{|A|}$ to every strict subset A, at all times $t \geq 0$;*

3. *an observable M satisfies $\mathrm{Tr}(MP) \neq 0$, that is, its expectation value depends on the slow amplitude, only if M acts nontrivially on every site.*

Proof. The partial trace of a product operator factorizes, and $\mathrm{Tr}\, Z_j = 0$ for every traced site, so $\mathrm{Tr}_{\bar{A}}\, P = 0$ whenever $\bar{A} \neq \emptyset$. Items (1) and (2) follow by linearity, using Corollary 2 for the time dependence. For (3), if M acts as the identity on some site j, then $M = M' \otimes \mathbf{1}_j$ and $\mathrm{Tr}(MP) = \mathrm{Tr}(M'\, \mathrm{Tr}_j\, P) = 0$. □

5 The Mpemba crossing

Let $0 < \alpha < \beta < 1$ and

$$\rho_P = \frac{\mathbf{1} + \alpha P}{2^n}, \qquad \rho_Q = \frac{\mathbf{1} + \beta X_n}{2^n}.$$

Both are states: P and X_n are Hermitian with eigenvalues ± 1, so the eigenvalues of ρ_P and ρ_Q are $(1 \pm \alpha)/2^n$ and $(1 \pm \beta)/2^n$, each with multiplicity 2^{n-1}. Write $\rho_P(t) = e^{t\mathcal{L}}\rho_P$, $\rho_Q(t) = e^{t\mathcal{L}}\rho_Q$, and let $T(\rho) = \frac{1}{2}\|\rho - \mathbf{1}/2^n\|_1$ denote the trace distance to equilibrium.

Theorem 4. *For all $t \geq 0$,*

$$T(\rho_P(t)) = \tfrac{1}{2}\alpha e^{-2\epsilon t}, \qquad T(\rho_Q(t)) = \tfrac{1}{2}\beta e^{-2t}.$$

The state ρ_Q starts farther from equilibrium and is strictly closer to it for all $t > t^*$, where

$$t^* = \frac{\log(\beta/\alpha)}{2 - 2\epsilon}.$$

The same crossing, at the same time t^* and with the same reversal of the order afterwards, holds for every distance to $\mathbf{1}/2^n$ of the form $h(\rho)$ with h unitarily invariant and strictly increasing along $a \mapsto (\mathbf{1} + aU)/2^n$, $a \in [0, 1)$, for a Pauli string $U \neq \mathbf{1}$; this includes the Hilbert–Schmidt distance and the relative entropy $S(\rho \,\|\, \mathbf{1}/2^n)$. Throughout, every strict reduction of $\rho_P(t)$ is maximally mixed, whereas the reduction of $\rho_Q(t)$ to site n is $(\mathbf{1} + \beta e^{-2t} X)/2$.

Proof. By Proposition 1, $\mathcal{L}(P) = -2\epsilon P$ and $\mathcal{L}(X_n) = -2X_n$ (for X_n: $k = 1$, $e = 0$, $f_1 = 0$ since $n \geq 2$), and $\mathcal{L}(\mathbf{1}) = 0$. Hence $\rho_P(t) = (\mathbf{1} + a_P(t)P)/2^n$ and $\rho_Q(t) = (\mathbf{1} + a_Q(t)X_n)/2^n$ with $a_P(t) = \alpha e^{-2\epsilon t}$ and $a_Q(t) = \beta e^{-2t}$. For a Hermitian unitary $U \neq \mathbf{1}$ that is a Pauli string, $\|aU\|_1 = |a|\, 2^n$, which gives the trace distances. The difference $\log a_Q(t) - \log a_P(t) = \log(\beta/\alpha) - (2 - 2\epsilon)t$ is strictly decreasing, positive at $t = 0$, and vanishes at t^*, so $a_Q(t) < a_P(t)$ exactly for $t > t^*$. Any two Pauli strings different from $\mathbf{1}$ are unitarily conjugate, so $\rho_P(t)$ and $\rho_Q(t)$ are unitarily conjugate states of the family $(\mathbf{1} + aU)/2^n$ evaluated at $a_P(t)$ and $a_Q(t)$; a unitarily invariant h that is strictly increasing in a therefore orders them exactly as a_P and a_Q do. For the Hilbert–Schmidt distance, $\|aU/2^n\|_2 = a\, 2^{-n/2}$. For the relative entropy, $S((\mathbf{1} + aU)/2^n \,\|\, \mathbf{1}/2^n) = \frac{1}{2}\big[(1 + a)\log(1 + a) + (1 - a)\log(1 - a)\big]$, whose derivative $\frac{1}{2}\log\frac{1+a}{1-a}$ is positive for $0 < a < 1$. The statements on reductions are Proposition 3 for ρ_P and a direct partial trace for ρ_Q. □

For example, with $\epsilon = 1/6$, $\alpha = 0.3$ and $\beta = 0.8$, the crossing time is $t^* = \frac{3}{5}\log\frac{8}{3} \approx 0.5885$.

The theorem makes the separation quantitative. A monitor that sees only strict subsystems declares $\rho_P(t)$ at equilibrium for all t, and sees $\rho_Q(t)$ out of equilibrium on the last site; it orders the two states opposite to their late-time global order and misses the existence of the slow tail altogether. The slow tail of ρ_P is real nonetheless: its trace distance to equilibrium decays only as $e^{-2\epsilon t}$, and for $\epsilon \to 0$ its relaxation time $1/(2\epsilon)$ diverges.

Remark 5 (single-observable witnesses)**.** In the framework of [3], an observable M witnesses the crossing through its coefficients on the slow and fast eigenmodes, and the construction uses a nonzero coefficient on the slow mode. For the generator (1) the slow mode is P, so the witness needs $\mathrm{Tr}(MP) \neq 0$, and by Proposition 3(3) it acts nontrivially on all n sites. This is a statement about supports, not about the difficulty of the measurement: $\mathrm{Tr}(P\rho)$ is the mean of the product of the outcomes of single-site Z measurements on one copy of ρ, so local measurements with correlated classical post-processing estimate it.

6 Verification

The Python checker assembles $\mathcal{L}$ as a $4^n \times 4^n$ matrix for $n = 2, 3, 4, 5$ and $\epsilon \in \{1/6, 1/2, 9/10\}$, diagonalizes it numerically, and compares the result, with multiplicities, to the rate formula and to the closed form of Proposition 1, which it evaluates in rational arithmetic. It checks Corollary 2 exactly from the rates. For $n \leq 4$ it evolves ρ_P and ρ_Q with the matrix exponential of $\mathcal{L}$, checks the trace distances of Theorem 4, locates the crossing by root finding and compares it with t^*, checks that $\rho_P(t^*)$ and $\rho_Q(t^*)$ are isospectral, and checks that every strict reduction of $\rho_P(t)$ is maximally mixed. Its recorded output lists the exact spectra. All deviations are below 10^{-9}, and the checker runs in about ten seconds.

7 Scope and limitations

The proofs are analytic and cover every $n \geq 2$ and every $0 < \epsilon < 1$. The numerical checks confirm the conventions and the closed forms for $n \leq 5$; they are not a formal verification.

The obstruction concerns observables and reduced states supported on strict subsets of sites, measured on a single copy. It does not say that the slow amplitude is hard to measure: products of local outcomes give it directly, as noted in Remark 5. The construction is a single explicit family; it does not characterize which local Lindbladians have a slowest mode invisible to strict reductions, nor does it address thermal (non-unital) dynamics, where the stationary state is not maximally mixed.

On bibliographic priority, the note relies on the works cited above and no systematic search for antecedents was documented. The individual ingredients, Pauli-diagonal dissipators and global parity as a weakly broken conservation law, are standard, and we make no claim of priority for the construction.

References

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[3] J. M. Z. Choquehuanca, P. A. C. Obando, M. S. Sarandy and F. M. de Paula, *Witnessing the quantum Mpemba effect with a single observable*, preprint (2026), arXiv:2607.12163.