Momentum Rényi and Tsallis entropies behind hard walls: the existence threshold $1/(2(m+1))$

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Abstract

A particle confined by impenetrable walls has a wave function that vanishes outside its well. Cutting the wave function off at a wall produces an algebraic tail in the momentum distribution, and Rényi or Tsallis entropies of order $\alpha < 1$, which weigh tails heavily, exist only when α exceeds a threshold α_{TH}. Olendski found $\alpha_{\mathrm{TH}} = 1/4$ for Dirichlet walls and $\alpha_{\mathrm{TH}} = 1/2$ for Neumann and Robin walls in the one-dimensional well, and conjectured that $1/2$ is the threshold of every one-dimensional non-Dirichlet system. We prove a general criterion: if the first derivative of the zero-extended wave function that jumps has order m, the momentum entropies exist exactly for $\alpha > \alpha_{\mathrm{TH}} = 1/(2(m+1))$, and the Rényi entropy diverges logarithmically at the threshold with an explicit rate. For eigenfunctions of a Schrödinger operator with integrable potential between hard walls only $m = 0$ and $m = 1$ occur, so $\alpha_{\mathrm{TH}} = 1/4$ when both walls are Dirichlet and $\alpha_{\mathrm{TH}} = 1/2$ otherwise. This proves Olendski's conjecture for every level, for real or complex Robin lengths, and for any integrable potential inside the well; the same holds on a half-line. The underlying mechanism, that endpoint jumps govern the decay of a Fourier integral, is classical; the contribution is its conversion into a two-sided existence criterion, which requires controlling the interference between two walls, and the divergence rate.

1 Introduction

A quantum particle in a well with impenetrable walls has a wave function that is zero beyond the walls. Its momentum distribution is the squared modulus of the Fourier transform of this truncated function, and the truncation costs a slowly decaying tail at large momentum. Information-theoretic measures of the momentum distribution feel this tail. For a normalised $\psi \in L^2(\mathbb{R})$ let

$$\phi(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) e^{-ipx}\, dx, \qquad \gamma(p) = |\phi(p)|^2, \qquad I_\alpha = \int_{\mathbb{R}} \gamma(p)^\alpha\, dp.$$

The momentum Rényi and Tsallis entropies, in the normalisation of [1], are

$$R_\gamma(\alpha) = \frac{1}{1-\alpha} \ln I_\alpha, \qquad T_\gamma(\alpha) = \frac{1}{\alpha - 1}(1 - I_\alpha),$$

and for $\alpha \neq 1$ they exist exactly when $I_\alpha < \infty$. When $\alpha < 1$ the power γ^α enhances the tail, so the entropies exist only above a threshold α_{TH} fixed by its decay rate. Orders just above $1/2$ are physically relevant: in the Rényi entropic uncertainty relation of Bialynicki-Birula [5], the momentum order α and the position order β satisfy $1/\alpha + 1/\beta = 2$, so α ranges over $(1/2, \infty)$ and tends to $1/2$ as $\beta \to \infty$.

Olendski computed these entropies for the one-dimensional well of width a [1]. From the explicit momentum wave functions he found $\alpha_{\mathrm{TH}} = 1/4$ for the Dirichlet condition $\psi = 0$ at the walls and $\alpha_{\mathrm{TH}} = 1/2$ for the Neumann condition $\psi' = 0$, by a comparison test on the explicit formulas. Noting that the same value $1/2$ appears for the attractive Robin wall and for mixed Dirichlet–Neumann walls, he conjectured that "the threshold α_{TH} of one half is characteristic of any 1D non-Dirichlet system" [1]. The class of systems is not specified and no proof is given.

The simplest example shows the mechanism. The Neumann ground state $\psi = a^{-1/2}$ on $[0, a]$ jumps from 0 to $a^{-1/2}$ at each wall, so $|\phi(p)|$ decays like $1/|p|$ and $\gamma^\alpha \sim |p|^{-2\alpha}$ is integrable exactly for $\alpha > 1/2$. The Dirichlet ground state is continuous, its derivative jumps, $|\phi|$ decays like $|p|^{-2}$, and the threshold is $1/4$. The general statement measures the order m of the first derivative that jumps.

Theorem 4 shows that, if the zero-extended wave function has its first jump in the derivative of order m, then $I_\alpha < \infty$ if and only if $\alpha > 1/(2(m+1))$. The delicate direction is divergence at and below the threshold: with two walls, the leading coefficient of the momentum tail is a trigonometric sum that may vanish periodically, and Lemma 3 shows that it stays large on a set of momenta of positive density. Proposition 5 gives the exact divergence rate of I_α as $\alpha \downarrow \alpha_{\mathrm{TH}}$. Corollary 6 applies the theorem to eigenfunctions of Schrödinger operators with integrable potential and proves Olendski's conjecture in that setting.

2 Related work

Olendski's paper [1] determines the thresholds $1/4$ and $1/2$ for the Dirichlet and Neumann wells by a comparison test on explicit formulas, and states the conjecture. It writes the Dirichlet range as the closed interval $[1/4, \infty)$, and its introduction states that the momentum entropies exist "at $\alpha \geq 1/2$ only", while its section on the Neumann well uses a strict inequality; Theorem 4 shows that the strict inequality is correct for both thresholds. The attractive Robin wall on the half-line had been treated earlier in closed form [2]. Before that, Aptekarev, Dehesa, Sánchez-Moreno and Tulyakov [4] computed the momentum entropic moments of every level of the Dirichlet well in closed form for integer orders, where convergence is automatic, and left non-integer orders open.

Olendski later extended the observation case by case to other geometries. For a Dirichlet annulus he found the value $2/5$, independent of the inner radius and of the level [3], in line with a threshold fixed by the behaviour at the boundary; that paper does not analyse how this threshold depends on the Aharonov–Bohm flux it introduces. All these thresholds are obtained from explicit wave functions. Section 6 explains how wall-singular potentials and higher dimensions would fit the formula $1/(2(m+1))$ heuristically.

The mechanism behind Lemma 2 is classical: the large-frequency asymptotics of a Fourier integral over an interval is governed by the jumps of derivatives at the endpoints, as follows from repeated integration by parts. What the present note adds is the full translation into an existence criterion for the momentum entropies, including the divergence direction, which requires controlling the interference between the contributions of two walls, and the divergence rate at the threshold.

3 Setting and statement

Hypothesis 1 (H_m)**.** Let $m \geq 0$ be an integer. The function ψ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. If $m \geq 1$, ψ is of class C^{m-1} on $\mathbb{R}$, $\psi^{(m-1)}$ is absolutely continuous, and $\psi', \ldots, \psi^{(m)}$ are

integrable on $\mathbb{R}$. There are points $x_1 < \cdots < x_N$, $N \geq 1$, such that on each open interval of $\mathbb{R} \setminus \{x_1, \ldots, x_N\}$ the function $\psi^{(m)}$ is locally absolutely continuous, and its derivative $h = \psi^{(m+1)}$, defined almost everywhere, is integrable on $\mathbb{R}$. The jumps $J_k = \psi^{(m)}(x_k^+) - \psi^{(m)}(x_k^-)$ are all nonzero.

The one-sided limits exist because h is integrable, and for the same reason, together with the integrability of $\psi^{(m)}$, $\psi^{(m)}$ tends to zero at $\pm\infty$. The integer m is the order of the first derivative of ψ that jumps. For a smooth function on $[a, b]$ extended by zero, $N \leq 2$, the points are the walls, m is the smallest order such that $\psi^{(m)}(a)$ or $\psi^{(m)}(b)$ is nonzero, and the jumps are $\psi^{(m)}(a)$ at a and $-\psi^{(m)}(b)$ at b (only the nonzero ones are kept). Set

$$G(p) = \sum_{k=1}^{N} J_k e^{-ipx_k}, \qquad \sigma^2 = \sum_k |J_k|^2, \qquad S = \sum_k |J_k|.$$

Lemma 2 (leading term)**.** *Under (H_m), for every $p \neq 0$,*

$$\sqrt{2\pi}\,(ip)^{m+1}\phi(p) = G(p) + \hat{h}(p), \qquad \hat{h}(p) = \int_{\mathbb{R}} h(x)e^{-ipx}\,dx \xrightarrow[|p|\to\infty]{} 0.$$

Proof. For $j \leq m$, j integrations by parts on $\mathbb{R}$ give $\int_{\mathbb{R}} \psi^{(j)} e^{-ipx} dx = (ip)^j \sqrt{2\pi}\,\phi(p)$; there are no boundary terms because $\psi, \ldots, \psi^{(m-1)}$ are absolutely continuous, integrable with integrable derivatives, and hence tend to zero at infinity. On each interval of $\mathbb{R}\setminus\{x_k\}$ one more integration by parts gives

$$\int \psi^{(m)} e^{-ipx} dx = \Big[\frac{\psi^{(m)} e^{-ipx}}{-ip}\Big] + \frac{1}{ip}\int h e^{-ipx} dx.$$

Summing over the intervals, the boundary terms at x_k combine into $J_k e^{-ipx_k}/(ip)$ and those at infinity vanish, so $(ip)^m\sqrt{2\pi}\,\phi(p) = \big(G(p) + \hat{h}(p)\big)/(ip)$. The Riemann–Lebesgue lemma gives $\hat{h}(p) \to 0$. □

Lemma 3 (the leading coefficient is large on a set of positive density)**.** *If $N \geq 2$, let $\delta = \min_{k\neq l} |x_k - x_l|$, $T_0 = 4S^2/(\sigma^2\delta)$ and $\mu_0 = 1/\delta$. For every real P, the set $\Omega_P = \{p \in [P, P + T_0] : |G(p)| \geq \sigma/2\}$ has Lebesgue measure at least μ_0. If $N = 1$, $|G|$ is constant and equal to σ, and the statement holds with $T_0 = \mu_0 = 1$.*

Proof. Expanding the square,

$$\frac{1}{T_0}\int_P^{P+T_0} |G|^2 dp = \sigma^2 + \sum_{k\neq l} J_k \bar{J}_l \,\frac{1}{T_0}\int_P^{P+T_0} e^{-ip(x_k - x_l)} dp.$$

Each oscillatory integral is bounded in modulus by $2/(T_0\delta)$, so the cross terms are bounded by $2S^2/(T_0\delta) = \sigma^2/2$, and $\int_P^{P+T_0} |G|^2 dp \geq T_0\sigma^2/2$. Since $|G| \leq S$ everywhere and $|G| < \sigma/2$ outside Ω_P, also $\int_P^{P+T_0} |G|^2 dp \leq S^2|\Omega_P| + T_0\sigma^2/4$. Hence $|\Omega_P| \geq T_0\sigma^2/(4S^2) = 1/\delta$. □

Lemma 3 is where the two walls interfere. The coefficient G can vanish: $G(p) = \psi(a)e^{-ipa} - \psi(b)e^{-ipb}$ has periodic zeros when $|\psi(a)| = |\psi(b)|$. It cannot, however, be small on a proportion of momenta large enough to restore convergence.

Theorem 4 (threshold)**.** *Under (H_m) and for $\alpha > 0$,*

$$I_\alpha = \int_{\mathbb{R}} \gamma(p)^\alpha\, dp < \infty \quad \Longleftrightarrow \quad \alpha > \alpha_{\mathrm{TH}} = \frac{1}{2(m+1)}.$$

The momentum entropies $R_\gamma(\alpha)$ and $T_\gamma(\alpha)$ therefore exist for $\alpha > \alpha_{\mathrm{TH}}$, $\alpha \neq 1$, and do not exist for $0 < \alpha \leq \alpha_{\mathrm{TH}}$.

Proof. The function ϕ is continuous and bounded by $\|\psi\|_1/\sqrt{2\pi}$, so γ^α is locally integrable. By Lemma 2 there is $P_1 \geq 1$ such that $|\hat{h}(p)| \leq \sigma/4$ for $|p| \geq P_1$. For such p, $\gamma(p) \leq (S+\sigma/4)^2/(2\pi|p|^{2m+2})$, so $I_\alpha < \infty$ as soon as $2\alpha(m+1) > 1$. Conversely, for $P \geq P_1$ and $p \in \Omega_P$ one has $|G(p) + \hat{h}(p)| \geq \sigma/4$, hence $\gamma(p) \geq \sigma^2/(32\pi p^{2m+2})$ and

$$\int_P^{P+T_0} \gamma^\alpha \, dp \geq \mu_0 \Big(\frac{\sigma^2}{32\pi}\Big)^\alpha (P+T_0)^{-2\alpha(m+1)}.$$

Summing over $P = P_1 + jT_0$, $j \geq 0$, gives a series whose general term is equivalent to a constant times $j^{-2\alpha(m+1)}$, which diverges when $2\alpha(m+1) \leq 1$. □

Proposition 5 (divergence rate)**.** *Assume (H_m) with $N \leq 2$, and let M be the mean value of the periodic function $|G|^{1/(m+1)}$: $M = |J_1|^{1/(m+1)}$ if $N = 1$, and*

$$M = \frac{1}{2\pi} \int_0^{2\pi} \big(|J_1|^2 + |J_2|^2 + 2|J_1||J_2| \cos\theta\big)^{\frac{1}{2(m+1)}} d\theta$$

if $N = 2$. Then

$$\lim_{\alpha \downarrow \alpha_{\mathrm{TH}}} (\alpha - \alpha_{\mathrm{TH}})\, I_\alpha = \frac{(2\pi)^{-\alpha_{\mathrm{TH}}} M}{m+1} > 0, \qquad R_\gamma(\alpha) = \frac{1}{1-\alpha_{\mathrm{TH}}} \ln \frac{1}{\alpha - \alpha_{\mathrm{TH}}} + O(1).$$

Proof. We use an Abelian fact: if $f : [P, \infty) \to [0, U]$ is measurable, $P \geq 1$, and $\frac{1}{T}\int_P^{P+T} f \to M$ as $T \to \infty$, then $(s-1)\int_P^\infty f(p) p^{-s} dp \to M$ as $s \downarrow 1$. Indeed, with $F(p) = \int_P^p f \leq Up$, an integration by parts gives $\int_P^\infty f p^{-s} dp = s \int_P^\infty F(p) p^{-s-1} dp$. For $\varepsilon > 0$ there is P_ε with $|F(p) - Mp| \leq \varepsilon p$ for $p \geq P_\varepsilon$, whence

$$\Big| s \int_P^\infty F p^{-s-1} dp - sM \int_P^\infty p^{-s} dp \Big| \leq s(U+M)P_\varepsilon + \frac{s\varepsilon}{s-1};$$

multiplying by $s-1$ and letting $s \to 1$, then $\varepsilon \to 0$, proves the claim.

Set $s = 2\alpha(m+1)$, so $s - 1 = 2(m+1)(\alpha - \alpha_{\mathrm{TH}})$. For $p \geq P_1$, Lemma 2 reads $\gamma(p)^\alpha = (2\pi)^{-\alpha} p^{-s} f_\alpha(p)$ with $f_\alpha = |G + \hat{h}|^{2\alpha}$. Since $\big||G + \hat{h}| - |G|\big| \leq |\hat{h}| \to 0$ and $(u, \alpha) \mapsto u^{2\alpha}$ is uniformly continuous on $[0, 2S] \times [\alpha_{\mathrm{TH}}, 1]$, for every $\eta > 0$ there are $P_\eta \geq P_1$ and $\alpha_\eta > \alpha_{\mathrm{TH}}$ such that $|f_\alpha(p) - |G(p)|^{2\alpha_{\mathrm{TH}}}| \leq \eta$ for $p \geq P_\eta$ and $\alpha_{\mathrm{TH}} < \alpha \leq \alpha_\eta$; the corresponding error on $(s-1)\int_{P_\eta}^\infty f_\alpha p^{-s} dp$ is at most $\eta P_\eta^{1-s} \leq \eta$. For $N \leq 2$, $|G|^{2\alpha_{\mathrm{TH}}} = |G|^{1/(m+1)}$ is periodic (constant if $N = 1$) with mean M, so the Abelian fact gives the limit M on the positive half-line. The same holds on the negative half-line, since $|G(-p)|^2 = |J_1|^2 + |J_2|^2 + 2|J_1||J_2|\cos(\cdot)$ has the same mean. The contribution of $|p| \leq P_\eta$ is bounded uniformly in α and disappears after multiplication by $s - 1$. Altogether $(s-1)I_\alpha \to 2(2\pi)^{-\alpha_{\mathrm{TH}}} M$, which is the stated limit; $M > 0$ by Lemma 3. The expansion of R_γ follows by taking logarithms. □

The divergence of the Rényi entropy at the threshold is therefore logarithmic, as Olendski observes in the text of [1] (Section 3 and Conclusions), with the universal coefficient $1/(1 - \alpha_{\mathrm{TH}})$: 2 for a non-Dirichlet wall and 4/3 for the Dirichlet well. For the Neumann ground state of the well of width a, $\psi = a^{-1/2}$, $J_1 = -J_2 = a^{-1/2}$, $M = 4/(\pi\sqrt{a})$, and

$$\lim_{\alpha \downarrow 1/2} \Big(\alpha - \frac{1}{2}\Big) I_\alpha = \frac{4}{\pi\sqrt{2\pi a}} \approx \frac{0.5079}{\sqrt{a}}.$$

For $N \geq 3$ the proposition should remain true with M the Bohr mean value of the almost periodic function $|G|^{1/(m+1)}$; this extension is not written out here.

4 Hard-wall eigenfunctions: the conjecture

Corollary 6 (well with hard walls)**.** *Let $a < b$, let $V \in L^1(a,b)$ be a complex-valued potential and $E \in \mathbb{C}$, and let ψ be a solution of $-\psi'' + V\psi = E\psi$ on $[a,b]$, not identically zero, with ψ and ψ' absolutely continuous, extended by zero outside $[a,b]$. If $\psi(a) = \psi(b) = 0$, then $\alpha_{\mathrm{TH}} = 1/4$; in every other case $\alpha_{\mathrm{TH}} = 1/2$. In particular, if one of the walls carries a Robin condition $\psi'(a) = \psi(a)/\Lambda_a$ or $-\psi'(b) = \psi(b)/\Lambda_b$ with $\Lambda \in \mathbb{C} \setminus \{0\}$, or a Neumann condition ($\Lambda = \infty$), then every eigenfunction, at every level, has $\alpha_{\mathrm{TH}} = 1/2$.*

Proof. By the uniqueness theorem for linear differential equations with integrable coefficients, a solution with $\psi(c) = \psi'(c) = 0$ at some $c \in [a,b]$ vanishes identically. If $\psi(a) \neq 0$ or $\psi(b) \neq 0$, the extended function satisfies (H_0): it is bounded with compact support, its derivative $h = \psi'$ on (a,b) is integrable, and its jumps are $\psi(a)$ at a and $-\psi(b)$ at b, of which only the nonzero ones are kept; where ψ vanishes at a wall the extension is continuous there. If $\psi(a) = \psi(b) = 0$, the extended function is absolutely continuous on $\mathbb{R}$, its derivative jumps by $\psi'(a) \neq 0$ at a and by $-\psi'(b) \neq 0$ at b by uniqueness, and $h = \psi'' = (V - E)\psi$ is integrable on (a,b) because ψ is bounded: this is (H_1). Theorem 4 concludes. Finally, at a Robin or Neumann wall, $\psi(a) = 0$ would force $\psi'(a) = 0$ and hence $\psi \equiv 0$, so $\psi(a) \neq 0$. □

Corollary 7 (half-line)**.** *Let ψ be a nonzero solution of the same equation on $[0,\infty)$, with V locally integrable on $[0,\infty)$, extended by zero for $x < 0$, such that ψ, ψ' and ψ'' are integrable on $(0,\infty)$. If $\psi(0) \neq 0$, then $\alpha_{\mathrm{TH}} = 1/2$; if $\psi(0) = 0$ and $\psi'(0) \neq 0$, then $\alpha_{\mathrm{TH}} = 1/4$.*

Proof. Hypotheses (H_0) and (H_1) are verified as in Corollary 6, with $N = 1$; $\psi \in L^2$ because ψ is integrable and bounded. □

The attractive Robin wall, with bound state $\psi(x) = \sqrt{2\kappa}\, e^{-\kappa x}$ on the half-line [2], falls under the first case. Here $\gamma(p) = \kappa/(\pi(\kappa^2 + p^2))$ and

$$I_\alpha = \Big(\frac{\kappa}{\pi}\Big)^{\alpha} \kappa^{1-2\alpha} \sqrt{\pi}\, \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)},$$

finite exactly for $\alpha > 1/2$, with a simple pole of residue $\sqrt{\kappa/\pi} = (2\pi)^{-1/2}|J_1|$, as Proposition 5 predicts. The quasi-one-dimensional hydrogen ground state, $\psi(x) \propto xe^{-x}$, satisfies (H_1) directly with a single jump of ψ' at the origin and recovers the threshold $1/4$ reported by Olendski.

The theorem also explains why only two values occur in these models: an eigenfunction of a Schrödinger operator with integrable potential cannot have both ψ and ψ' zero at a wall, so $m \in \{0,1\}$. Values $m \geq 2$ require functions that are not such eigenfunctions, for example $\psi \propto x^2(1-x)^2$ on $[0,1]$, whose second derivative jumps by 2 at 0 and by -2 at 1, so that $\alpha_{\mathrm{TH}} = 1/6$.

5 Verification

The statements above are proved analytically. The public Python checker illustrates them numerically and writes its verification record to the same entry. It builds nine eigenfunctions of the well $[0,1]$ with Neumann, Dirichlet, mixed, real Robin ($\Lambda = 0.5$, -0.3, 2, 0.7) and complex Robin ($\Lambda = 0.4 + 0.3i$) walls, whose Fourier transforms are known in closed form and are checked against quadrature to about 10^{-16}. It verifies that $m = 1$ exactly when both walls are Dirichlet; that the difference between $|p|^{m+1}\sqrt{2\pi}\,|\phi(p)|$ and $|G(p)|$, multiplied by P, stays bounded on windows at $\pm P$ for $P = 10^2, 10^3, 10^4$ (Lemma 2); that the dyadic

blocks $\int_P^{2P} \gamma^{\alpha_{\mathrm{TH}}} dp$ approach the nonzero constant $\ln 2\,(2\pi)^{-\alpha_{\mathrm{TH}}} M$, within a relative 2×10^{-3} at $P = 6400$, which is the divergence at the threshold; and that the ratio of successive blocks equals $2^{1-2\alpha(m+1)}$ within 5×10^{-3} for $\alpha = \alpha_{\mathrm{TH}} \pm 0.1$. It also treats the polynomial example with $m = 2$, the closed form for the attractive Robin wall, and the divergence rate of the Neumann ground state through a Hurwitz zeta representation: $(\alpha - \frac{1}{2})I_\alpha = 0.5131$, 0.5085 and 0.5080 for $\alpha - \frac{1}{2} = 10^{-2}, 10^{-3}, 10^{-4}$, against the limit 0.5079. The checker requires NumPy, SciPy and mpmath and runs in under a minute. These computations illustrate the proof and do not replace it.

6 Scope and limitations

Analytic proof. Lemmas 2 and 3, Theorem 4, Proposition 5 for $N \leq 2$ and Corollaries 6 and 7. The conjecture is thus established for a particle on a bounded interval or a half-line, with a wave function vanishing beyond hard walls and solving a Schrödinger equation with integrable potential (locally integrable near the wall on the half-line), with at least one wall where the condition is not Dirichlet. It holds for every level, for real or complex Robin lengths, and also for superpositions and non-stationary states whenever ψ does not vanish at a wall and satisfies (H_0).

Not covered. A one-dimensional system without hard walls has no threshold of this type, and "non-Dirichlet" has no meaning there. Potentials that are singular at the wall, such as $\mathfrak{a}/x^2$, produce $\psi \sim x^\nu$ with non-integer ν, where $\nu = \frac{1}{2}(1 + \sqrt{1 + 4\mathfrak{a}})$; the natural extension of Theorem 4 predicts the threshold $1/(2(\nu + 1)) = 1/(3 + \sqrt{1 + 4\mathfrak{a}})$, but this fractional case is not proved here. In the same spirit, for the singular "volcano" potential of [3] the threshold is controlled by the azimuthal quantum number and the Aharonov–Bohm flux. In dimension d, the heuristic $\gamma \sim |p|^{-(d+1+2m)}$ integrated against $|p|^{d-1}d|p|$ gives $\alpha_{\mathrm{TH}} = d/(d + 1 + 2m)$, that is $d/(d+1)$ for Neumann and $d/(d+3)$ for Dirichlet conditions; no general proof is attempted for arbitrary domains.

Bibliographic priority. The particular thresholds are due to Olendski: $1/4$ and $1/2$ for the Dirichlet, Neumann, mixed and Robin wells [1], $1/2$ for the attractive Robin wall on the half-line [2], and the value for annuli [3]. The asymptotic expansion of Lemma 2 is classical. The general criterion in terms of m, with the divergence direction for two walls, the divergence rate, and the dichotomy of Corollary 6 for every integrable potential and every real or complex Robin length, were not found in the works read for this note: [1, 2, 3, 4, 5], all read in full. The twelve works citing [1] listed by OpenAlex and Semantic Scholar (2020 to 2026) were examined by title or abstract, and targeted web searches on momentum Rényi entropies, boundary conditions and Fourier decay returned no general statement. This search does not establish priority.

References

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