How much leakage makes the Margenau–Hill state of a post-selected system positive

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Abstract

A quantum system observed between a preparation and a later post-selection is described by two objects: the state ρ inferred from the past and the effect E of the future outcome. A natural candidate for its state at the intermediate time is the Margenau–Hill combination $(E\rho + \rho E)/2$, normalized by the outcome probability, but this operator can have negative eigenvalues. Kam and Wong showed that this happens whenever the post-selection forbids a subspace that is not invariant under ρ. Real post-selections are imperfect: a forbidden subspace leaks with some small probability η. We determine exactly how much leakage restores positivity. For the effect $E = P + \eta(\mathbf{1} - P)$, where P is an orthogonal projector, write ρ in blocks A, C, D relative to P and let $\kappa = \|A^{+1/2} C D^{+1/2}\|$ be the largest normalized coherence between the two blocks, with Moore–Penrose inverses. Then $\kappa \le 1$, and the Margenau–Hill operator is positive semidefinite if and only if

$$\eta \;\ge\; \eta_{\mathrm{crit}} = \frac{1 - \sqrt{1-\kappa^2}}{1 + \sqrt{1-\kappa^2}} = \frac{\kappa^2}{4} + \frac{\kappa^4}{8} + O(\kappa^6),$$

in any finite dimension. Below the threshold we give an explicit vector on which the operator is negative. The threshold is below one exactly when $\operatorname{rank}\rho = \operatorname{rank} A + \operatorname{rank} D$, so for a pure state with weight in both blocks no leakage short of $\eta = 1$ suffices. For a qubit with Bloch vector $(4/5, 0, 0)$ and $P = |0\rangle\langle 0|$, the threshold is exactly $1/4$.

1 Introduction

A system that is prepared, left to evolve, and then post-selected on a later measurement outcome carries information from both ends of the time interval. Quantum state smoothing [1] asks for the best estimate of the state at an intermediate time given both a past and a future measurement record. In the simplest description the past is summarized by a density matrix ρ, the filtered state, and the future by an effect $0 \le E \le \mathbf{1}$, the retrofiltered effect. Combining the two into a single operator that behaves like a state is not straightforward, because ρ and E need not commute.

Kam and Wong [2] study the family of orderings $\Phi_s(E,\rho) = \frac{1}{2}\big(\rho^s E \rho^{1-s} + \rho^{1-s} E \rho^s\big)$, $s \in [0,1]$. All members have trace $\operatorname{Tr}(E\rho)$, the probability of the post-selected outcome. The symmetric member $\rho^{1/2} E \rho^{1/2}$ is always positive. The end point

$$\Phi_0(E,\rho) = \tfrac{1}{2}(E\rho + \rho E)$$

is the Margenau–Hill ordering [2], and it can fail to be positive. Their Theorem 2 states that if $\Phi_0(E,\rho) \ge 0$ then $\ker E$ is invariant under ρ; equivalently, a forbidden subspace $\ker E$ that is not ρ-invariant gives Φ_0 a negative eigenvalue.

A post-selection that forbids a subspace exactly is an idealization. In practice the forbidden subspace leaks: states in it pass the post-selection with some small probability η. The simplest model is the effect

$$E_\eta = P + \eta(\mathbf{1} - P), \qquad 0 \le \eta \le 1,$$

where P is an orthogonal projector; for $\eta = 0$ the forbidden subspace is $\mathrm{ran}(\mathbf{1} - P)$, and for $\eta = 1$ there is no post-selection at all and $\Phi_0 = \rho$. This note answers the question: what is the smallest leakage η for which the Margenau–Hill operator $\Phi_0(E_\eta, \rho)$ is positive? Since dividing by $\mathrm{Tr}(E_\eta \rho) > 0$ does not affect positivity, the same threshold governs the normalized Margenau–Hill state.

The answer depends on ρ through a single number. Writing ρ in blocks $A = P\rho P$, $C = P\rho(\mathbf{1} - P)$, $D = (\mathbf{1} - P)\rho(\mathbf{1} - P)$, the quantity $\kappa = \|A^{+1/2} C D^{+1/2}\|$ measures the largest coherence between the kept and the leaking subspaces, relative to their populations. We show that $\kappa \le 1$ and that $\Phi_0(E_\eta, \rho) \ge 0$ exactly when $4\eta/(1+\eta)^2 \ge \kappa^2$, that is, when η exceeds $\eta_{\mathrm{crit}} = (1 - \sqrt{1-\kappa^2})/(1 + \sqrt{1-\kappa^2})$. Weak coherence needs leakage of order $\kappa^2/4$; maximal coherence $\kappa = 1$, which occurs for every pure state with weight in both subspaces, cannot be compensated by any leakage below one.

A single qubit already displays the whole phenomenon. Take $\rho = \frac{1}{2}(\mathbf{1} + \frac{4}{5}X)$ and $P = |0\rangle\langle 0|$. The Margenau–Hill operator is $\Phi_0 = \frac{1}{2}\begin{pmatrix} 1 & \frac{2}{5}(1+\eta) \\ \frac{2}{5}(1+\eta) & \eta \end{pmatrix}$, with determinant $-(4\eta - 1)(\eta - 4)/100$. It is negative for $\eta < 1/4$ and positive for $1/4 \le \eta \le 1$, and indeed $\kappa = 4/5$ gives $\eta_{\mathrm{crit}} = 1/4$. The general result replaces the off-diagonal entry by the matrix $A^{+1/2} C D^{+1/2}$ and its modulus by the operator norm.

2 Related work

Guevara and Wiseman [1] formulated quantum state smoothing, the estimation of a quantum state from past and future records, which is the setting in which operators built from a filtered state ρ and a retrofiltered effect E arise.

The question answered here is a quantitative refinement of the work of Kam and Wong [2]. Their Theorem 1 shows, for full-rank ρ and an effect with nontrivial kernel, that an open interval of orderings Φ_s around $s = 1/2$ remains positive if and only if $\ker E$ is ρ-invariant, and their Theorem 2 shows that the Margenau–Hill member Φ_0 cannot be positive when $\ker E$ is not ρ-invariant. Both statements are qualitative and concern effects with a kernel. A leaking effect E_η with $\eta > 0$ has trivial kernel, so their theorems do not apply to it, and the present note measures how far the obstruction persists. At $\eta = 0$ our criterion reduces to $C = 0$, which is precisely the invariance of $\ker E_0 = \mathrm{ran}(\mathbf{1} - P)$ under ρ; for this family of effects it thus recovers their Theorem 2 together with its converse (Remark 4).

The proofs use the Schur-complement characterization of positive semidefinite block matrices with a possibly singular diagonal block. It is classical; we include the short proof in Lemma 1 so that the note is self-contained.

3 Setting

Let $\mathcal{H}$ be a finite-dimensional Hilbert space, $\rho \ge 0$ a density matrix on $\mathcal{H}$, and P an orthogonal projector with $0 \ne P \ne \mathbf{1}$. Relative to the decomposition $\mathcal{H} = \mathrm{ran}\, P \oplus \mathrm{ran}(\mathbf{1} - P)$ write

$$\rho = \begin{pmatrix} A & C \\ C^\dagger & D \end{pmatrix}, \qquad E_\eta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \eta\mathbf{1} \end{pmatrix}.$$

A direct multiplication gives

$$\Phi(\eta) := \Phi_0(E_\eta, \rho) = \tfrac{1}{2}(E_\eta \rho + \rho E_\eta) = \begin{pmatrix} A & \beta C \\ \beta C^\dagger & \eta D \end{pmatrix}, \qquad \beta = \frac{1+\eta}{2}. \tag{1}$$

For a positive semidefinite matrix M we write M^+ for its Moore–Penrose inverse, $M^{+1/2} = (M^{1/2})^+$, and $\Pi_M = MM^+$ for the orthogonal projector onto $\operatorname{ran} M$. Define

$$K = A^{+1/2}\, C\, D^{+1/2}, \qquad \kappa = \|K\|,$$

the operator norm. When A and D are invertible, $K = A^{-1/2} C D^{-1/2}$ and κ is the largest singular value of the coherence block after each diagonal block has been normalized to the identity.

Lemma 1 (Schur complement)**.** *Let $M = \left(\begin{smallmatrix} A & B \\ B^\dagger & F \end{smallmatrix}\right)$ be Hermitian with $A \geq 0$. Then $M \geq 0$ if and only if $\operatorname{ran} B \subseteq \operatorname{ran} A$ and $F - B^\dagger A^+ B \geq 0$. In that case*

$$M = L^\dagger \begin{pmatrix} A & 0 \\ 0 & F - B^\dagger A^+ B \end{pmatrix} L, \qquad L = \begin{pmatrix} \mathbf{1} & A^+ B \\ 0 & \mathbf{1} \end{pmatrix}, \tag{2}$$

and for every vector v the vector $u = -A^+ B v$ satisfies $\langle (u,v), M(u,v)\rangle = \langle v, (F - B^\dagger A^+ B) v\rangle$.

Proof. Suppose $M \geq 0$ and let $x \in \ker A$. For a real t and any v, $\langle (tx, v), M(tx, v)\rangle = 2t \operatorname{Re}\langle x, Bv\rangle + \langle v, Fv\rangle \geq 0$, which forces $\langle x, Bv\rangle = 0$. Hence $\ker A \subseteq \ker B^\dagger$, that is, $\operatorname{ran} B \subseteq \operatorname{ran} A$. Whenever this inclusion holds, $AA^+B = B$, and expanding $L^\dagger \operatorname{diag}(A, F - B^\dagger A^+ B)\, L$ gives back M, which is (2). Since L is invertible, $M \geq 0$ is then equivalent to $F - B^\dagger A^+ B \geq 0$. The last identity follows by expanding $\langle u, Au\rangle + 2\operatorname{Re}\langle u, Bv\rangle + \langle v, Fv\rangle$ with $u = -A^+Bv$ and $A^+AA^+ = A^+$. □

Lemma 2 (Factorization of the coherence)**.** *If $\rho \geq 0$ then $\operatorname{ran} C \subseteq \operatorname{ran} A$ and $\operatorname{ran} C^\dagger \subseteq \operatorname{ran} D$. Consequently $C = A^{1/2} K D^{1/2}$, $C^\dagger A^+ C = D^{1/2} K^\dagger K D^{1/2}$, and $K^\dagger K = \Pi_D K^\dagger K \Pi_D$.*

Proof. The first inclusion is Lemma 1 applied to ρ; the second is the same lemma applied to ρ with the two blocks interchanged. The inclusions give $\Pi_A C = C = C \Pi_D$. Since $A^{1/2} A^{+1/2} = \Pi_A$ and $D^{+1/2} D^{1/2} = \Pi_D$, we get $A^{1/2} K D^{1/2} = \Pi_A C \Pi_D = C$. Then $C^\dagger A^+ C = D^{1/2} K^\dagger A^{1/2} A^+ A^{1/2} K D^{1/2} = D^{1/2} K^\dagger \Pi_A K D^{1/2}$, and $\Pi_A K = K$ because $\operatorname{ran} A^{+1/2} = \operatorname{ran} A$. Finally $K \Pi_D = K$ because $D^{+1/2} \Pi_D = D^{+1/2}$, which gives the last identity. □

4 The threshold

Theorem 3. *Let $\rho \geq 0$, P and κ be as above. Then $\kappa \leq 1$, and for $0 \leq \eta \leq 1$ the Margenau–Hill operator $\Phi(\eta)$ of (1) is positive semidefinite if and only if*

$$\frac{4\eta}{(1+\eta)^2} \;\geq\; \kappa^2, \tag{3}$$

that is, if and only if $\eta \geq \eta_{\mathrm{crit}}$ with

$$\eta_{\mathrm{crit}} = \frac{1 - \sqrt{1-\kappa^2}}{1 + \sqrt{1-\kappa^2}}.$$

Proof. If $D = 0$ then $C = 0$ by Lemma 2, so $\kappa = 0$, $\eta_{\mathrm{crit}} = 0$, and $\Phi(\eta) = \mathrm{diag}(A, 0) \geq 0$ for every η; the statement holds. Assume $D \neq 0$. Apply Lemma 1 to $\Phi(\eta)$ with $B = \beta C$ and $F = \eta D$. The range condition $\mathrm{ran}(\beta C) \subseteq \mathrm{ran}\, A$ holds by Lemma 2, so $\Phi(\eta) \geq 0$ if and only if

$$S_\eta := \eta D - \beta^2 C^\dagger A^+ C = D^{1/2}(\eta\, \Pi_D - \beta^2 K^\dagger K) D^{1/2} \;\geq\; 0,$$

where the factorization uses Lemma 2 and $D^{1/2}\Pi_D D^{1/2} = D$. The operator $\eta\Pi_D - \beta^2 K^\dagger K$ is supported on $\mathrm{ran}\, D$, on which $D^{1/2}$ is invertible, so $S_\eta \geq 0$ if and only if $\eta\Pi_D \geq \beta^2 K^\dagger K$. As $K^\dagger K$ is supported on $\mathrm{ran}\, D$ and its largest eigenvalue is κ^2, this holds if and only if $\eta \geq \beta^2\kappa^2$, which is (3).

For $\eta = 1$ we have $\Phi(1) = \rho \geq 0$, and (3) gives $\kappa \leq 1$. The function $f(\eta) = 4\eta/(1+\eta)^2$ is continuous and strictly increasing on $[0, 1]$, since $f'(\eta) = 4(1-\eta)/(1+\eta)^3 > 0$ there, with $f(0) = 0$ and $f(1) = 1$. The set of $\eta \in [0, 1]$ satisfying (3) is therefore an interval $[\eta_*, 1]$, where η_* is the unique solution of $f(\eta) = \kappa^2$ in $[0, 1]$. Writing $\sigma = \sqrt{1-\kappa^2}$, the equation $\kappa^2(1+\eta)^2 = 4\eta$ has the two roots $(1-\sigma)/(1+\sigma)$ and $(1+\sigma)/(1-\sigma)$, whose product is 1; the first lies in $[0, 1]$, so $\eta_* = \eta_{\mathrm{crit}}$. (For $\kappa = 0$ the criterion holds for every η and $\eta_{\mathrm{crit}} = 0$.) □

Three features of the threshold are immediate. It depends on ρ and P only through κ, and not on the dimension. It increases from 0 at $\kappa = 0$ to 1 at $\kappa = 1$, with expansion

$$\eta_{\mathrm{crit}} = \frac{\kappa^2}{4} + \frac{\kappa^4}{8} + \frac{5\kappa^6}{64} + O(\kappa^8),$$

so weak coherence is compensated by leakage of order $\kappa^2/4$. And the positive set is the whole interval $[\eta_{\mathrm{crit}}, 1]$: more leakage never destroys positivity once it has been reached.

Remark 4. At $\eta = 0$ the criterion reads $\kappa = 0$, which by Lemma 2 is equivalent to $C = 0$, that is, to the invariance of $\ker E_0 = \mathrm{ran}(\mathbf{1} - P)$ under ρ. For this family of effects the theorem therefore contains Theorem 2 of [2] and its converse: $\Phi_0(E_0, \rho) \geq 0$ if and only if $\ker E_0$ is ρ-invariant. The note does not require ρ to have full rank.

5 The negative direction below threshold

The proof of Theorem 3 also produces an explicit vector on which $\Phi(\eta)$ is negative when $\eta < \eta_{\mathrm{crit}}$. Assume $\kappa > 0$, and let y be a unit right singular vector of K for its largest singular value, so that $\|Ky\| = \kappa$. Since $y \in \mathrm{ran}\, K^\dagger \subseteq \mathrm{ran}\, D$, the vector $v = D^{+1/2}y$ satisfies $\langle v, Dv\rangle = \|\Pi_D y\|^2 = 1$. Put

$$u = -\beta A^+ C v, \qquad w = (u, v) \in \mathrm{ran}\, P \oplus \mathrm{ran}(\mathbf{1} - P).$$

Proposition 5. *For every $\eta \in [0, 1]$,*

$$\langle w, \Phi(\eta) w\rangle = \eta - \frac{(1+\eta)^2}{4}\,\kappa^2 = \frac{(1+\eta)^2}{4}\,(f(\eta) - \kappa^2),$$

which is negative exactly when $\eta < \eta_{\mathrm{crit}}$.

Proof. By the last statement of Lemma 1 with $B = \beta C$ and $F = \eta D$, $\langle w, \Phi(\eta) w\rangle = \langle v, S_\eta v\rangle$. With the factorization of S_η, $\langle v, S_\eta v\rangle = \langle y, \Pi_D D^{1/2} D^{+1/2}(\eta\Pi_D - \beta^2 K^\dagger K) D^{+1/2} D^{1/2}\Pi_D y\rangle = \eta\|y\|^2 - \beta^2\|Ky\|^2 = \eta - \beta^2\kappa^2$, because $D^{1/2}D^{+1/2} = \Pi_D$ and $\Pi_D y = y$. The sign statement is Theorem 3. □

The witness has a transparent structure. Its component v in the leaking subspace is the direction of maximal normalized coherence, and its component u in the kept subspace is the best linear response to v, which is how the Schur complement isolates the off-diagonal contribution. The depth of the negativity, $\eta - \beta^2\kappa^2$, is largest at $\eta = 0$, where it equals $-\kappa^2/4$ for this normalization of w.

6 When the threshold is below one

Proposition 6. $\kappa < 1$ *if and only if* $\operatorname{rank}\rho = \operatorname{rank}A + \operatorname{rank}D$. *In particular* $\kappa < 1$ *whenever* ρ *has full rank, and* $\kappa = 1$ *for every pure state* $\rho = |\psi\rangle\langle\psi|$ *with* $P|\psi\rangle \neq 0$ *and* $(\mathbf{1}-P)|\psi\rangle \neq 0$.

Proof. If $D = 0$ then $\kappa = 0$ and $\operatorname{rank}\rho = \operatorname{rank}A$, since $C = 0$. Otherwise, applying (2) to ρ itself shows that ρ is congruent to $\operatorname{diag}(A, S_1)$ with $S_1 = D^{1/2}(\Pi_D - K^\dagger K)D^{1/2}$, so $\operatorname{rank}\rho = \operatorname{rank}A + \operatorname{rank}S_1$. Since $0 \le K^\dagger K \le \Pi_D$ and $D^{1/2}$ is invertible on $\operatorname{ran}D$, $\operatorname{rank}S_1 = \operatorname{rank}D$ exactly when $\Pi_D - K^\dagger K$ is invertible on $\operatorname{ran}D$, that is, when $\kappa < 1$. For full-rank ρ the blocks A and D are invertible and the ranks add. For the pure state, A and D have rank one while ρ has rank one, so $\kappa = 1$. □

The second case has a direct physical reading: if the system is in a pure state with amplitude in both the kept and the leaking subspace, the Margenau–Hill operator of any genuinely selective leaky post-selection, $\eta < 1$, has a negative eigenvalue. A leakage below one can restore positivity only when the ranks of the two diagonal blocks add up to the rank of ρ, which is the case for every full-rank state.

7 Qubit example

Example 7. Let $\rho = \frac{1}{2}(\mathbf{1} + xX + yY + zZ)$ with $x^2+y^2+z^2 \le 1$ and $|z| < 1$, and $P = |0\rangle\langle 0|$. Then $A = (1+z)/2$, $D = (1-z)/2$ and $C = (x-iy)/2$, so

$$\kappa^2 = \frac{x^2+y^2}{1-z^2}, \qquad \det\Phi(\eta) = \frac{\eta(1-z^2) - \beta^2(x^2+y^2)}{4}.$$

The condition $\kappa < 1$ says that the Bloch vector lies strictly inside the sphere, in agreement with Proposition 6. For $(x,y,z) = (4/5,0,0)$ one finds $\kappa = 4/5$, $\sqrt{1-\kappa^2} = 3/5$ and $\eta_{\mathrm{crit}} = (2/5)/(8/5) = 1/4$; the determinant equals $-(4\eta-1)(\eta-4)/100$, which vanishes at $\eta = 1/4$ and at the reciprocal root $\eta = 4$. For $(x,y,z) = (3/5,0,0)$ one finds $\eta_{\mathrm{crit}} = 1/9$. In this example the explicit witness of Proposition 5 is $v = (1-z)^{-1/2}\sqrt{2}$ and $u = -\beta(x-iy)\,v/(1+z)$, up to a phase.

8 Verification

The results above are proved analytically. The public Python checker tests them independently and writes its recorded output into the same entry. In exact symbolic arithmetic it confirms that η_{crit} is the smaller root of $\kappa^2(1+\eta)^2 = 4\eta$, that the two roots are reciprocal, the series expansion of η_{crit}, and the determinant and thresholds of Example 7. In floating-point arithmetic it draws random density matrices of dimension 2 to 6 (full rank, rank deficient and pure) with random projectors of every rank, 900 instances in all, together with 200 instances in which both diagonal blocks are singular. For each it computes κ with Moore–Penrose inverses and checks that $\kappa \le 1$, that the smallest eigenvalue of $\Phi(\eta)$ is nonnegative on $[\eta_{\mathrm{crit}}, 1]$, that the witness of Proposition 5 takes the predicted value and that $\Phi(\eta)$ is negative below η_{crit}, that a bisection on the sign of the smallest eigenvalue reproduces η_{crit}, and that $\kappa < 1$ exactly when $\operatorname{rank}\rho = \operatorname{rank}A + \operatorname{rank}D$. The bisection agrees with the formula to 3×10^{-10} when $\kappa < 1$; when $\kappa = 1$ the smallest eigenvalue vanishes to second order at $\eta = 1$ and the agreement is 10^{-4}, while the witness confirms negativity for every tested $\eta < 1$. The checker uses a fixed seed and runs in a few seconds.

9 Scope and limitations

Theorem 3 and Propositions 5 and 6 are proved analytically in any finite dimension, for arbitrary density matrices including singular ones. The computations confirm these statements numerically and verify the exact examples; they are not a substitute for the proofs.

The result concerns the Margenau–Hill ordering $s = 0$ and the isotropic leakage model $E_\eta = P+\eta(\mathbf{1}-P)$. Two natural extensions are left open: the positivity of the intermediate orderings Φ_s with $0 < s < 1/2$ under leakage, which governs the interval of admissible orderings studied in [2], and anisotropic leakage, in which $\eta(\mathbf{1} - P)$ is replaced by a general effect F supported on $\mathrm{ran}(\mathbf{1} - P)$. In the latter case the lower-right block of Φ_0 becomes $(FD + DF)/2$ and the off-diagonal block $C(\mathbf{1} + F)/2$, and the reduction to a single scalar κ no longer applies.

On bibliographic priority, the threshold formula was not found in the documented search, which consisted of reading the work of Kam and Wong [2] and of Guevara and Wiseman [1]. The result is an elementary consequence of the Schur-complement criterion, and a related statement may exist elsewhere; this note does not certify priority.

References

[1] I. Guevara and H. Wiseman, *Quantum state smoothing*, Phys. Rev. Lett. **115**, 180407 (2015); doi:10.1103/PhysRevLett.115.180407; arXiv:1503.02799.

[2] C.-F. Kam and K.-W. Wong, *Forbidden subspaces in quantum state smoothing*, arXiv preprint (2026); arXiv:2609.17307.