Every real three-qubit state can be prepared with rotations and three CZ gates

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Abstract

A three-qubit state is real when its eight amplitudes in the computational basis are real numbers. Such states can be prepared from $|000\rangle$ by a circuit that is itself real: single-qubit rotations $R_y(\theta)$ and controlled-Z (CZ) gates between pairs of qubits. The two-qubit gates are the expensive resource, and the question is how many CZ gates the worst real state requires. Perdomo, Castaneda and Vogeler proved that four always suffice, that three suffice when Cayley's hyperdeterminant of the amplitudes is nonnegative, and conjectured that four is optimal. We show that the conjecture is false. Every real three-qubit state is prepared by R_y rotations and the three gates CZ_{01}, CZ_{12}, CZ_{01}, so a linear nearest-neighbour architecture suffices, and three is optimal. The proof rests on a characterization of the states reachable with two CZ gates: they are exactly the real states for which one of the three correlations $\langle Y_iY_k\rangle$ vanishes, that is, for which some pair of qubits has zero rebit concurrence. This yields the exact CZ count of every real three-qubit state. For example, the W state needs exactly three CZ gates, and the state $\frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$, whose hyperdeterminant equals $-1/4$, is prepared with three CZ gates by an explicit circuit with angles that are multiples of $\pi/2$.

1 Introduction

Preparing a prescribed quantum state from a product state is the first step of many quantum algorithms, and on current hardware the cost of a preparation circuit is dominated by its two-qubit gates. For three qubits with arbitrary complex single-qubit gates the answer is classical: three CNOT gates suffice to prepare any pure state [1]. Many targets of practical interest are real, that is, all their amplitudes in the computational basis are real numbers, and it is natural to prepare them with a circuit that stays in the real numbers throughout: rotations

$$R_y(\theta) = e^{-i\theta Y/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

on single qubits, and the controlled-Z gate $\mathrm{CZ} = \mathrm{diag}(1, 1, 1, -1)$ on any pair of qubits. Real local gates carry half as many parameters as complex ones, so nothing guarantees a priori that three two-qubit gates still suffice.

Perdomo, Castaneda and Vogeler [2] studied exactly this setting. For a real state $|\phi\rangle = \sum_{n=0}^{7} w_n|n\rangle$ they consider

$$\Delta(\phi) = (w_0w_7 - w_1w_6 - w_2w_5 + w_3w_4)^2 - 4\,(w_1w_2 - w_0w_3)(w_5w_6 - w_4w_7), \tag{1}$$

which is Cayley's hyperdeterminant of the $2\times2\times2$ tensor of amplitudes [7]. They prove that every real three-qubit state is prepared with R_y rotations and at most four CZ gates, that

three CZ gates suffice when $\Delta(\phi) \geq 0$, and they conjecture that four is optimal, that is, that some real three-qubit states cannot be prepared with real local gates and fewer than four CZ gates.

This note shows that the conjecture is false. Three CZ gates suffice for every real three-qubit state, in the fixed arrangement $\mathrm{CZ}_{01}, \mathrm{CZ}_{12}, \mathrm{CZ}_{01}$ on a line of three qubits, and three is the optimal worst-case count. The mechanism rests on two observations. First, the states reachable with two CZ gates are exactly those for which one correlation $E_{ik} = \langle Y_i Y_k \rangle$ between two qubits vanishes, where Y is the Pauli matrix; $|E_{ik}|$ is the rebit concurrence of the reduced state of qubits i and k [4], so the condition says that this pair is unentangled. Second, conjugating $Y_0 Y_2$ by CZ_{01} gives $Y_0 Z_1 Y_2$, and a single rotation of the middle qubit always cancels the expectation value of this operator, because it turns Z_1 within the plane spanned by Z_1 and X_1. The three correlations E_{01}, E_{02}, E_{12} thus determine the exact CZ count of every real three-qubit state, and the sign of the hyperdeterminant plays no role in it.

2 Related work

Žnidarič, Giraud and Georgeot [1] showed that any three-qubit pure state can be prepared from a product state with at most three CNOT gates when single-qubit gates are arbitrary unitaries, and that any three-qubit state can be transformed into any other with at most four. Since CNOT and CZ differ by Hadamard gates on the target, the same counts hold for CZ gates. Their construction uses complex local gates, and the present note shows that the same count of three is achieved when local gates are restricted to real rotations. Their class of states reachable with two CNOT gates, of the form $\cos\phi|\alpha\beta\gamma\rangle + \sin\phi|\alpha^{\perp}\beta'\gamma'\rangle$, is the complex counterpart of the set characterized in Theorem 4, where real local gates reduce membership to a single quadratic condition. They also observe that on a line of three qubits, where only nearest neighbours interact, their construction needs one additional CNOT for the GHZ-type states of their class 3; they state no lower bound for this architecture. For real states, Theorem 5 shows that three CZ gates on a line always suffice.

Perdomo, Castaneda and Vogeler [2] introduced the real setting studied here, proved the bound of four CZ gates and the sufficient condition $\Delta \geq 0$ for three, and stated the conjecture that four is optimal. They give no lower-bound argument and no example of a state claimed to need four gates. Our Theorem 5 refutes the conjecture, and Section 7 explains why their criterion is sufficient but not necessary. An earlier paper of Perdomo [3] reduces every real three-qubit state by local rotations to a canonical form with four parameters; apart from a remark in its introduction, it does not address CZ counts.

The quantities that govern the count come from the entanglement theory of rebits, the real-amplitude analogue of qubits. Caves, Fuchs and Rungta [4] showed that the entanglement of formation of a real two-qubit density matrix ρ is a function of the rebit concurrence $|\mathrm{Tr}(\rho\, Y \otimes Y)|$, which vanishes exactly when ρ is separable; E_{ik} is its signed version for the reduced state of qubits i and k. Wootters [7] defined the rebit three-tangle σ_{ABC}, showed that for a pure state it equals four times Cayley's hyperdeterminant without absolute value, so that it can be negative, and related it to the squared signed concurrences; the absolute value of the same quantity is the three-tangle of Coffman, Kundu and Wootters [5], $\tau_{ABC} = 4|\Delta|$. Section 7 uses this relation to compare the criterion $\Delta \geq 0$ with Theorem 4. Wootters had earlier exhibited a three-rebit state in which every pair is maximally entangled [6]; it is the example N of Section 6.

Real circuits have been studied mainly for unitaries. Wei and Di [8] showed that every two-qubit gate in $SO(4)$ is a real circuit with two CNOT gates and six R_y rotations, the unitary analogue of Lemma 3, and synthesized every gate in $SO(8)$ with sixteen CNOT gates and

single-qubit gates that include R_z rotations. These results concern gates; the CZ count of real three-qubit state preparation is not addressed there.

A recent classification discusses real-amplitude targets without settling the question. Lee and Kim [10] classify three-qubit states and give explicit preparation circuits with general single-qubit rotations; they cite [2] for its CZ-based formulation and its discussion of real targets but do not treat CZ counts for real circuits. Counting parameters gives a lower bound: a real n-qubit state depends on $2^n - 1$ real parameters, the first layer of rotations supplies n angles, and each CZ gate is followed by at most two new angles on the qubits it acts on, so at least $\lceil (2^n - n - 1)/2 \rceil$ CZ gates are needed, which equals 2 for $n = 3$. Theorem 4 below shows that this bound is not attained for generic states, so the true worst-case value for $n = 3$ is three. Real three-qubit state preparation also appears as a subroutine: Kristjuhan and Berry [9] prepare a state with real-valued amplitudes and implement a three-qubit block of their phase-estimation circuit as a three-qubit state preparation (Sec. IV), without stating its two-qubit gate count; Theorem 5 bounds that cost by three CZ gates.

3 Setting and preliminaries

Qubits are labelled $0, 1, 2$, and a subscript indicates the qubit on which an operator acts. For an angle a let

$$D(a) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} = R_y(2a), \qquad c(a) = D(a)|0\rangle = (\cos a, \sin a)^{\mathsf{T}}.$$

The allowed local gates are the real orthogonal 2×2 matrices, namely the rotations $D(a)$ and the reflections $D(a)Z$. Reflections add nothing: Z commutes with every CZ gate, satisfies $ZD(a) = D(-a)Z$ and fixes $|0\rangle$, so every reflection can be pushed to the beginning of the circuit, where it acts trivially. Circuits with orthogonal local gates and circuits with rotations therefore prepare the same states, and the global sign -1 is itself the rotation $D(\pi)$. All states below are real and normalized, and for such a state ψ and a pair $\{i, k\}$ of qubits we set

$$E_{ik}(\psi) = \langle \psi | Y_i Y_k | \psi \rangle.$$

The operator $Y_i Y_k$ is real and symmetric, so $E_{ik}(\psi)$ is a real quadratic form in the amplitudes of ψ. Its absolute value is the rebit concurrence of the reduced state ρ_{ik} [4].

Lemma 1 (Invariance)**.** *For every real density matrix ρ of one qubit, $\mathrm{Tr}(\rho Y) = 0$. The quantities E_{ik} are invariant under any rotation $D(a)$ applied to any qubit.*

Proof. The matrix Y is purely imaginary and antisymmetric, so its trace against a real symmetric matrix vanishes. The invariance follows because $D(a) = e^{-iaY}$ commutes with Y and with the identity. □

Lemma 2 (Slices)**.** *Let j be the qubit distinct from i and k, and write $\psi = |0\rangle_j \otimes T_0 + |1\rangle_j \otimes T_1$, where T_0 and T_1 are the real 2×2 matrices of amplitudes with rows indexed by qubit i and columns by qubit k. Then $E_{ik}(\psi) = -2(\det T_0 + \det T_1)$.*

Proof. With $\varepsilon = iY = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ we have $Y \otimes Y = -\varepsilon \otimes \varepsilon$, and for a real matrix T viewed as a vector of two qubits, $\sum_{a,b,a',b'} T_{ab}\, \varepsilon_{aa'} \varepsilon_{bb'}\, T_{a'b'} = 2(T_{00}T_{11} - T_{01}T_{10}) = 2 \det T$. Since $Y_i Y_k$ acts trivially on qubit j, the two slices contribute additively. □

Lemma 3 (Two qubits)**.** *Every real normalized state χ of two qubits is prepared from $|00\rangle$ with rotations and at most one CZ gate.*

Proof. Let M be the 2×2 matrix of amplitudes of χ. The real singular value decomposition gives $M = U\,\mathrm{diag}(s_0, s_1)\,V^{\mathsf{T}}$ with rotations U and V, $s_0 \geq 0$, $s_1 \in \mathbb{R}$ and $s_0^2 + s_1^2 = 1$; the sign of s_1 absorbs a possible determinant -1 of the orthogonal factors. Hence $\chi = (U \otimes V)(s_0|00\rangle + s_1|11\rangle)$. Choose μ with $\cos\mu = s_0$ and $\sin\mu = -s_1$. Since $c(\pi/4) = |+\rangle$,

$$\mathrm{CZ}\,(c(\mu) \otimes |+\rangle) = \cos\mu\,|0\rangle|+\rangle + \sin\mu\,|1\rangle|-\rangle,$$

and $D(-\pi/4)|+\rangle = |0\rangle$, $D(-\pi/4)|-\rangle = -|1\rangle$. Therefore

$$s_0|00\rangle + s_1|11\rangle = \big(I \otimes D(-\pi/4)\big)\,\mathrm{CZ}\,\big(D(\mu) \otimes D(\pi/4)\big)|00\rangle. \qquad \square$$

4 States reachable with two CZ gates

Theorem 4. *Let ψ be a real normalized three-qubit state and $\{i, j, k\} = \{0, 1, 2\}$. If $E_{ik}(\psi) = 0$, then ψ is prepared from $|000\rangle$ with rotations and the two gates CZ_{ij} followed by CZ_{jk}. Conversely, every state prepared with rotations and at most two CZ gates satisfies $E_{ik} = 0$ for at least one pair $\{i, k\}$.*

Proof. Sufficiency. Suppose $E_{ik}(\psi) = 0$ and let T_0, T_1 be the slices of Lemma 2, so that $\det T_0 + \det T_1 = 0$. The function $q(u) = \det(u_0 T_0 + u_1 T_1)$ on $u \in \mathbb{R}^2$ is a quadratic form whose symmetric matrix Q has trace $\det T_0 + \det T_1 = 0$. Its eigenvalues are therefore λ and $-\lambda$, with orthonormal eigenvectors e_+ and e_-. The vectors $u = (e_+ + e_-)/\sqrt{2}$ and $v = (e_+ - e_-)/\sqrt{2}$ are orthonormal and satisfy $q(u) = q(v) = (\lambda - \lambda)/2 = 0$. Replacing v by $-v$ if necessary, the matrix G with rows u and v is a rotation. The state $G_j\psi$ has slices $S_0 = u_0 T_0 + u_1 T_1$ and $S_1 = v_0 T_0 + v_1 T_1$, both singular, so $S_s = a_s\, c(\gamma_s)^{\mathsf{T}}$ with $a_s \in \mathbb{R}^2$ (qubit i) and an angle γ_s (qubit k), taking $a_s = 0$ if $S_s = 0$. Hence

$$G_j\psi = a_0 \otimes |0\rangle_j \otimes c(\gamma_0) + a_1 \otimes |1\rangle_j \otimes c(\gamma_1).$$

Put $\varrho = (\gamma_0 + \gamma_1)/2$, $\gamma = (\gamma_0 - \gamma_1)/2$ and $\chi = a_0 \otimes |0\rangle_j + a_1 \otimes |1\rangle_j$, a real normalized state of qubits i and j. Since $Z\,c(\gamma) = c(-\gamma)$,

$$\mathrm{CZ}_{jk}(\chi \otimes c(\gamma)) = a_0 \otimes |0\rangle \otimes c(\gamma) + a_1 \otimes |1\rangle \otimes c(-\gamma),$$

and $D_k(\varrho)$ maps $c(\gamma)$ to $c(\gamma_0)$ and $c(-\gamma)$ to $c(\gamma_1)$. Consequently $\psi = G_j^{-1} D_k(\varrho)\,\mathrm{CZ}_{jk}(\chi \otimes D(\gamma)|0\rangle)$, and χ is prepared with one gate CZ_{ij} by Lemma 3.

Necessity. A circuit without CZ gates produces a real product state, for which $E_{ik} = \langle Y_i\rangle\langle Y_k\rangle = 0$ by Lemma 1. If all CZ gates (one or two) act on the same pair $\{i, j\}$, qubit k remains in a real pure state uncorrelated with the others, and $E_{ik} = \mathrm{Tr}(\rho_i Y)\,\langle Y_k\rangle = 0$, where ρ_i is the real reduced state of qubit i. Otherwise the two gates act on two distinct pairs, which share exactly one qubit; call it j and name the other two so that the circuit applies CZ_{ij} and then CZ_{jk}. Immediately before the second gate the state is $\varphi = \chi_{ij} \otimes c_k$ with χ and c real, and the rotations after it do not change E_{ik} (Lemma 1). The relation $\mathrm{CZ}_{jk} Y_k \mathrm{CZ}_{jk} = Z_j Y_k$ and Lemma 1 give, for the final state ψ,

$$E_{ik}(\psi) = \langle\varphi|\mathrm{CZ}_{jk}\,Y_i Y_k\,\mathrm{CZ}_{jk}|\varphi\rangle = \langle\chi|Y_i Z_j|\chi\rangle\,\langle c|Y|c\rangle = 0. \qquad \square$$

The set of real states reachable with at most two CZ gates is therefore the union of the three quadric hypersurfaces $E_{01} = 0$, $E_{02} = 0$, $E_{12} = 0$ of the unit sphere $S^7 \subset \mathbb{R}^8$. None of the three quadratic forms vanishes identically (they all equal $2/3$ on the W state, see Section 6), so this set has measure zero.

5 Three CZ gates suffice

Theorem 5. *Every real normalized three-qubit state ψ is prepared from $|000\rangle$ with rotations and the three gates CZ_{01}, CZ_{12}, CZ_{01}, applied in this order. In particular a linear connectivity $0-1-2$ suffices.*

Proof. Let $e_Z = \langle\psi|Y_0Z_1Y_2|\psi\rangle$ and $e_X = \langle\psi|Y_0X_1Y_2|\psi\rangle$. A direct computation gives $D(\alpha)^{\mathsf{T}} Z\, D(\alpha) = \cos 2\alpha\, Z - \sin 2\alpha\, X$, so

$$\langle D_1(\alpha)\psi|\, Y_0Z_1Y_2\, |D_1(\alpha)\psi\rangle = e_Z \cos 2\alpha - e_X \sin 2\alpha.$$

This expression vanishes for some α: take 2α to be the argument of the vector (e_X, e_Z), and any α if $e_Z = e_X = 0$. Fix such an α and let $\varphi = \mathrm{CZ}_{01} D_1(\alpha)\psi$, a real normalized state. Since $\mathrm{CZ}_{01} Y_0 \mathrm{CZ}_{01} = Y_0Z_1$ and CZ_{01} commutes with Y_2,

$$E_{02}(\varphi) = \langle D_1(\alpha)\psi|\mathrm{CZ}_{01}\, Y_0Y_2\, \mathrm{CZ}_{01}|D_1(\alpha)\psi\rangle = \langle D_1(\alpha)\psi|Y_0Z_1Y_2|D_1(\alpha)\psi\rangle = 0.$$

Theorem 4 with $i=0$, $j=1$, $k=2$ prepares φ with rotations and the gates CZ_{01} then CZ_{12}. Finally $\psi = D_1(-\alpha)\,\mathrm{CZ}_{01}\,\varphi$. □

The proof is constructive and uses only expectation values, one diagonalization of a symmetric 2×2 matrix and three 2×2 singular value decompositions. After merging adjacent rotations, the circuit contains eight rotations: two on qubit 0, four on qubit 1 (the first being the fixed rotation $D(\pi/4)$) and two on qubit 2. It therefore carries seven free angles, as many as the dimension of the sphere S^7 of real normalized states.

Corollary 6 (Exact CZ count)**.** *The minimal number of CZ gates needed to prepare a real three-qubit state ψ from $|000\rangle$ with real local gates is*

- 0 *if ψ is a product state;*
- 1 *if ψ is a product across some bipartition $\{i,j\}|\{k\}$ but not a full product state;*
- 2 *if ψ is not a product across any bipartition and at least one of $E_{01}(\psi)$, $E_{02}(\psi)$, $E_{12}(\psi)$ vanishes;*
- 3 *in all other cases, which include almost every real state.*

In particular, three is the optimal worst-case number.

Proof. A circuit with at most one CZ gate, acting on $\{i,j\}$, produces a state of the form $\chi_{ij}\otimes c_k$, and Lemma 3 prepares every state of this form with at most one CZ gate. The last two cases follow from Theorems 4 and 5, and the measure statement from the remark after Theorem 4. □

6 Examples

The GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$ has $E_{01} = E_{02} = E_{12} = 0$ and is not a product across any bipartition, so it needs exactly two CZ gates. The W state $(|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ has $E_{01} = E_{02} = E_{12} = 2/3$ and needs exactly three; its hyperdeterminant vanishes, so it is already covered by the criterion $\Delta \geq 0$ of [2].

The state

$$N = \tfrac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$$

is the three-rebit state of Wootters in which every pair of qubits is maximally entangled [6, Eq. (13)]. It has $\Delta(N) = -1/4 < 0$, so it lies in the region where the method of [2] uses four CZ gates. Its correlations are $E_{01} = E_{02} = E_{12} = 1$, so by Corollary 6 it needs exactly three. The construction of Theorem 5 gives the following circuit, listed in order of application:

$$R_y(\tfrac{\pi}{2})_0,\ R_y(\tfrac{\pi}{2})_1,\ \mathrm{CZ}_{01},\ R_y(-\tfrac{\pi}{2})_1,\ R_y(-\tfrac{3\pi}{2})_0,\ R_y(\tfrac{\pi}{2})_2,\ \mathrm{CZ}_{12},\ R_y(\pi)_2,\ R_y(\tfrac{\pi}{2})_1,\ \mathrm{CZ}_{01}.$$

Its output is exactly N, as the checker verifies in exact arithmetic.

7 Relation to the hyperdeterminant criterion

The hyperdeterminant is multiplied by $(\det g_0 \det g_1 \det g_2)^2$ when the local gates $g_0 \otimes g_1 \otimes g_2$ act on the amplitude tensor, so it is invariant under real orthogonal local gates. The states with $\Delta < 0$ form a nonempty open subset of S^7 (it contains N), hence a set of positive measure; in a sample of 20 000 states drawn uniformly from S^7, 4 330, about 22%, have $\Delta < 0$. The criterion $\Delta \geq 0$ of [2] is therefore sufficient but not necessary for three CZ gates. The CZ count depends only on the vanishing of the three correlations E_{ik} and on product structure, and the sign of Δ does not enter it.

Wootters's monogamy relation for rebits makes the comparison explicit. For a pure real state, his Eqs. (14), (17) and (24) [7], with the one-qubit term $\sigma_{A|BC} = 4 \det \rho_A$, give

$$4\Delta(\psi) = 4 \det \rho_0 - E_{01}(\psi)^2 - E_{02}(\psi)^2,$$

where ρ_0 is the reduced state of qubit 0; the checker confirms this identity on random states. The criterion $\Delta \geq 0$ thus asks that the pairwise rebit concurrences of qubit 0 be small compared with its entanglement with the other two qubits, whereas Theorem 4 asks that one of them vanish. On N every concurrence equals 1 and $\det \rho_0 = 1/4$, so $4\Delta(N) = -1$.

The three-CZ circuit of [2] places one gate on each of the three pairs of qubits. The circuit of Theorem 5 uses the pair $\{0, 1\}$ twice and never the pair $\{0, 2\}$, and the proof relies on the identity $\mathrm{CZ}_{01} Y_0 Y_2 \mathrm{CZ}_{01} = Y_0 Z_1 Y_2$ attached to this arrangement. Whether the triangular arrangement alone reaches every real state is not settled here; the generic rank computation of Section 8 shows only that it reaches a full-dimensional set.

8 Verification

The theorems are proved above, and the public Python checker tests the construction independently; its recorded output is part of the same entry. The checker implements the synthesis of Theorems 4 and 5 and simulates the resulting circuits. It confirms the identities $\mathrm{CZ}_{01} Y_0 Y_2 \mathrm{CZ}_{01} = Y_0 Z_1 Y_2$ and $\mathrm{CZ}(Y \otimes Y)\mathrm{CZ} = X \otimes X$ exactly. It synthesizes a three-CZ circuit for 20 000 states drawn uniformly from S^7, and for 5 000 further states all with $\Delta < 0$; in both samples the largest infidelity $1 - |\langle \text{target} | \text{output} \rangle|$ is below 10^{-15}. It treats nine named or degenerate states (product, GHZ, W, N, three states that are products across a bipartition, $|{+}{+}{+}\rangle$ and a uniform superposition with one sign flipped), checks the converse of Theorem 4 on all nine ordered sequences of two CZ gates with 200 random circuits each, confirms numerically that Δ is invariant under orthogonal local gates, and checks the identity $4\Delta = 4 \det \rho_q - E_{qa}^2 - E_{qb}^2$ for each qubit q on 500 random states. It computes the generic rank of the Jacobian of the map from circuit angles to states: 6 for two-CZ chains, and, up to relabelling the qubits, 4 for the sequence $(01, 01, 01)$, 6 for $(01, 01, 02)$, and 7 for $(01, 02, 01)$, $(01, 02, 02)$ and $(01, 02, 12)$. Finally, in exact symbolic arithmetic, it verifies that the circuit of Section 6 outputs N, that $\Delta(N) = -1/4$, $E_{ik}(N) = 1$ and $E_{ik}(W) = 2/3$ for all pairs. The checker uses fixed seeds and runs in about ten seconds.

9 Scope and limitations

The analytic content is complete: Theorems 4 and 5 and Corollary 6 are proved above, and they show that the conjecture of [2] is false and that the optimal worst-case number of CZ gates for real three-qubit states with real local gates is three. The computations confirm the construction to machine precision on random and named states and verify the example circuit exactly; the validity of the result rests on the proofs, not on the samples. The fraction of states with $\Delta < 0$ is a numerical estimate.

The result concerns preparation from $|000\rangle$. It says nothing about the number of CZ gates needed to map one arbitrary real state to another by a real circuit; composing two preparations gives six, and the optimal value of that problem is not studied here. The case of four or more qubits is open: the parameter-counting bound $\lceil (2^n - n - 1)/2 \rceil$ equals 6 for $n = 4$, and this note gives no indication of whether it is attained.

On bibliographic priority, the identification of E_{ik} with the signed rebit concurrence and of 4Δ with the rebit three-tangle is due to [4, 7], and the two-CZ characterization, the three-CZ construction and the exact count are, to our knowledge, new. The search covered the five works citing [2] listed by Semantic Scholar, OpenAlex and INSPIRE as of 19 November 2026, of which only a textbook was not read, together with targeted searches on real circuit synthesis and on rebit entanglement; none of them treats the CZ count of real circuits. This does not certify priority. The discussion of [2] refers to its arXiv version.

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