Separable states with the low-order moments of states of arbitrary Schmidt rank at bounded energy

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Abstract

In continuous-variable experiments the entanglement of two light modes is usually assessed from the first and second moments of the field quadratures, the mean vector and the covariance matrix. We ask how much such data can say about the Schmidt number, the number of entangled degrees of freedom, when the mean photon number is bounded, and show that they cannot bound it uniformly. For every $r \geq 2$ we give a pure state of Schmidt rank r and a separable state with exactly the same first and second moments, and indeed the same third moments, whose common mean photon number is at most a prescribed bound independent of r. The two states are supported on the Fock vectors $|2j, 2j\rangle$, $0 \leq j < r$, the first as a coherent superposition and the second as the corresponding classical mixture. Replacing the spacing 2 by an integer m makes all quadrature moments of degree below $2m$ coincide, and a moment of degree $2m$ distinguishes the states, so the construction is sharp. At bounded energy the obstruction forces the r-th largest Schmidt coefficient to be small, which points to a better-posed question.

1 Introduction

Entanglement between two modes of light is most often certified from quadrature statistics. For Gaussian states the first and second moments determine the state, and criteria built on the covariance matrix detect entanglement and quantify it [1, 2]. A finer question concerns the Schmidt number [3], the smallest k such that a state is a mixture of pure states of Schmidt rank at most k, which counts how many degrees of freedom are genuinely entangled. Lower bounds on the Schmidt number of continuous-variable states can be sought from their first and second moments. It is natural to ask how far these moments can go: can a criterion based on the covariance matrix, perhaps complemented by a bound on the mean photon number, certify an arbitrarily large Schmidt number in a way that applies uniformly to all states?

The simplest example shows that it cannot. Take the two-mode states

$$|\psi_r\rangle = \sum_{j=0}^{r-1} \sqrt{p_j}\, |2j, 2j\rangle, \qquad \sigma_r = \sum_{j=0}^{r-1} p_j\, |2j, 2j\rangle\langle 2j, 2j|,$$

with weights $p_j > 0$ summing to 1. The first has Schmidt rank r, the second is a mixture of product states and is separable. A quadrature monomial of degree two changes the photon numbers by at most two in total, while passing from $|2j, 2j\rangle$ to $|2j', 2j'\rangle$ with $j \neq j'$ requires changing each photon number by at least two, four quanta in all. The off-diagonal coherences of $|\psi_r\rangle$, which carry its entanglement, are therefore invisible to every first and second moment,

and also to third moments. With geometric weights $p_j \propto z^j$ the mean photon number stays below $4z/(1-z)$ for every r.

The main result (Theorem 1 and Corollary 2) states the general form of this observation: for every degree d, every energy bound $E > 0$ and every $r \geq 2$, there is a pure state of Schmidt rank r and a separable state whose quadrature moments of degree at most d all coincide and whose common mean photon number is at most E. The construction uses the spacing m with $2m > d$ in place of 2, and Proposition 3 shows that this condition cannot be relaxed for these states. The obstruction concerns uniform identification of the Schmidt number from low-order moments. It does not prevent a particular covariance matrix from certifying a Schmidt number larger than one for a particular family of states. Proposition 4 shows that at bounded energy the obstruction necessarily involves a small Schmidt coefficient, and Section 5 formulates the more stable question that this suggests.

2 Related work

Duan, Giedke, Cirac and Zoller [1] and Simon [2] gave separability criteria for two modes in terms of second moments; for Gaussian states of two modes these criteria are necessary and sufficient, and for general states they are sufficient conditions for entanglement. Shchukin and Vogel [4] extended this approach to a hierarchy of conditions on moments of all orders, which together characterize the states with a non-positive partial transpose. Our example sits at the interface of these works: $|\psi_r\rangle$ has a non-positive partial transpose, but its moments of degree below $2m$ coincide with those of a separable state, so only conditions involving moments of degree at least $2m$ can detect it.

The Schmidt number of a mixed state was introduced by Terhal and Horodecki [3] as the smallest k such that the state is a mixture of pure states of Schmidt rank at most k. Any Schmidt-number criterion that depends on a state only through its first and second moments, possibly together with its mean photon number, takes the same value on $|\psi_r\rangle$ and on the separable σ_r, and therefore certifies no entanglement for $|\psi_r\rangle$; the result below makes this limitation uniform in r at a fixed energy. Criteria built on nonlinear functionals of the state, which are not determined by low-order quadrature moments, are not affected by this obstruction.

3 Results

Let $\hat{a}, \hat{b}$ be the annihilation operators of the two modes and $\hat{x}_A = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$, $\hat{p}_A = (\hat{a} - \hat{a}^\dagger)/(i\sqrt{2})$, and similarly $\hat{x}_B, \hat{p}_B$. A quadrature moment of degree ℓ of a state ρ is $\mathrm{Tr}(\rho \hat{W})$ for an ordered product $\hat{W}$ of ℓ operators taken among $\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B$. Symmetrized moments, the mean vector and the covariance matrix are linear combinations of these. All states considered here have finite support in the Fock basis, so every moment is finite. For integers $r \geq 2$ and $m \geq 1$ and weights $p_0, \ldots, p_{r-1} > 0$ with $\sum_j p_j = 1$, set

$$|\psi\rangle = \sum_{j=0}^{r-1} \sqrt{p_j}\, |mj, mj\rangle, \qquad \sigma = \sum_{j=0}^{r-1} p_j\, |mj, mj\rangle\langle mj, mj|.$$

Theorem 1. *The state $|\psi\rangle$ has Schmidt rank r, and σ is separable. Every quadrature moment of degree $\ell < 2m$ takes the same value on $|\psi\rangle\langle\psi|$ and on σ. The mean total photon number of both states equals $2m \sum_j j\, p_j$.*

Proof. The vectors $|mj\rangle$ are orthonormal on each side, so the given expansion of $|\psi\rangle$ is a Schmidt decomposition with r nonzero coefficients, and σ is a convex combination of product

states. Since $\hat{x}_A, \hat{p}_A$ are linear combinations of $\hat{a}, \hat{a}^\dagger$, and similarly on B, an ordered product $\hat{W}$ of ℓ quadratures is a linear combination of ordered products of ℓ operators among $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$. Such a product maps $|n_A, n_B\rangle$ to a multiple of $|n_A + s_A, n_B + s_B\rangle$ with $|s_A| + |s_B| \le \ell$. For $j \neq j'$ the matrix element $\langle mj, mj|\hat{W}|mj', mj'\rangle$ can be nonzero only if $s_A = s_B = m(j - j')$, which requires $2m|j - j'| \le \ell$ and is impossible when $\ell < 2m$. Therefore

$$\langle\psi|\hat{W}|\psi\rangle = \sum_{j,j'} \sqrt{p_j p_{j'}}\, \langle mj, mj|\hat{W}|mj', mj'\rangle = \sum_j p_j \langle mj, mj|\hat{W}|mj, mj\rangle = \mathrm{Tr}(\sigma\hat{W}).$$

The total photon number $\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$ is diagonal in the Fock basis, so the same computation applies to it and gives $\sum_j p_j \cdot 2mj$ for both states. □

For $m = 2$ the theorem says that the mean vectors, the covariance matrices and all third moments of $|\psi\rangle$ and σ coincide; the mean vectors vanish. The energy is the mean photon number up to the vacuum contribution, and any Hamiltonian quadratic in the quadratures also has the same expectation in both states.

Corollary 2. *Let $d \ge 1$ be an integer and $E > 0$. For every $r \ge 2$ there are a pure state of Schmidt rank r and a separable state whose quadrature moments of degree at most d all coincide and whose common mean photon number is at most E.*

Proof. Choose $m = \lfloor d/2 \rfloor + 1$, so that $2m > d$, and $z \in (0, 1)$ with $2mz/(1 - z) \le E$. Take $p_j = z^j / \sum_{i<r} z^i$ in Theorem 1. The mean $\mu_r = \sum_{j<r} j\, p_j$ of this truncated geometric law increases with r, because passing from r to $r + 1$ adds the value r, which exceeds every value already present, so $\mu_r \le \lim_{s\to\infty} \mu_s = z/(1 - z)$. The mean photon number $2m\mu_r$ is therefore at most $2mz/(1 - z) \le E$. □

In particular, for second moments ($d = 2$, $m = 2$) the bound is $4z/(1 - z)$, and any positive energy budget admits pairs of this type with arbitrarily large Schmidt rank. Any test whose input consists of quadrature moments of degree at most d and of an upper bound on the mean photon number assigns the same verdict to the two states, and since one of them is separable, it cannot certify any entanglement of the other.

Proposition 3 (Sharpness)**.** *For the states of Theorem 1, the moment $\langle\hat{a}^m\hat{b}^m\rangle$, a polynomial of degree $2m$ in the quadratures, equals $\sum_{j=0}^{r-2} \sqrt{p_j p_{j+1}}\, \frac{(m(j+1))!}{(mj)!} > 0$ for $|\psi\rangle$ and 0 for σ. In particular, for spacing $m = 1$ the second moment $\langle\hat{a}\hat{b}\rangle$ already distinguishes the two states.*

Proof. Since $\hat{a}^m|n\rangle = \sqrt{n!/(n - m)!}\,|n - m\rangle$, the operator $\hat{a}^m\hat{b}^m$ maps $|m(j + 1), m(j + 1)\rangle$ to $\frac{(m(j+1))!}{(mj)!}|mj, mj\rangle$ and maps $|0, 0\rangle$ to zero. Its diagonal elements on the vectors $|mj, mj\rangle$ vanish, which gives 0 for σ, and the off-diagonal elements between consecutive vectors give the stated sum for $|\psi\rangle$. The operator is a polynomial of degree $2m$ in the quadratures because $\hat{a} = (\hat{x}_A + i\hat{p}_A)/\sqrt{2}$ and $\hat{b} = (\hat{x}_B + i\hat{p}_B)/\sqrt{2}$. □

At bounded energy, the uniform obstruction always rests on a small Schmidt coefficient.

Proposition 4. *Let $|\phi\rangle$ be a pure two-mode state with mean total photon number at most E, and let $\lambda_1 \ge \lambda_2 \ge \cdots$ be the eigenvalues of its reduced states, the squared Schmidt coefficients. Then $\lambda_r \le E/(r(r - 1))$ for every $r \ge 2$.*

Proof. The mean photon number of mode A is $\mathrm{Tr}(\rho_A\hat{a}^\dagger\hat{a})$, where $\rho_A = \sum_i \lambda_i |e_i\rangle\langle e_i|$ with orthonormal eigenvectors e_i. Write $\hat{n} = \hat{a}^\dagger\hat{a}$ and $c_i = \langle e_i|\hat{n}|e_i\rangle \ge 0$. By Ky Fan's minimum principle, the sum of the diagonal values of $\hat{n}$ on any s orthonormal vectors is at least the sum

of its s smallest eigenvalues, so $C_s = \sum_{i\le s} c_i \ge \sum_{i\le s}(i-1)$ for every s. For every S, Abel summation with the nonnegative differences $\lambda_s - \lambda_{s+1}$ gives

$$\sum_{i\le S} \lambda_i c_i = \sum_{s<S} (\lambda_s - \lambda_{s+1}) C_s + \lambda_S C_S \ge \sum_{i\le S} \lambda_i (i-1),$$

and letting S grow yields $\mathrm{Tr}(\rho_A \hat{n}) = \sum_i \lambda_i c_i \ge \sum_i \lambda_i (i-1)$. The same bound holds for mode B, whose reduced state has the same nonzero eigenvalues. Hence $E \ge 2\sum_{i\le r} \lambda_i (i-1) \ge 2\lambda_r \sum_{i\le r}(i-1) = \lambda_r\, r(r-1)$. □

For the states of Corollary 2 the smallest nonzero eigenvalue of the reduced state is in fact $p_{r-1} \propto z^{r-1}$, exponentially small in r.

4 Verification

The results are proved above. The public Python checker confirms them on examples, and its recorded output is part of the same entry. It builds the quadrature operators as Fock matrices truncated at a level that the words considered cannot reach, so that no truncation error occurs, and evaluates every ordered word of length at most d in $\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B$ on $|\psi\rangle$ and on σ: for spacing $m = 2$, degree $d = 3$ and $2 \le r \le 6$, and for $m \in \{3, 4\}$, $d = 2m - 1$ and $2 \le r \le 5$, each with geometric and with random weights. The largest difference is below 3×10^{-13}, at the level of rounding. It also checks the value of $\langle \hat{a}^m \hat{b}^m \rangle$ in Proposition 3 for $m \le 4$, the difference of second moments for spacing 1, the Schmidt rank of $|\psi\rangle$, the equality of mean photon numbers, and the bound $2mz/(1-z)$ for $r \le 60$, and it tests Proposition 4 on 300 random pure states with finite Fock support. The bound of Proposition 4 is attained by the maximally entangled state $r^{-1/2} \sum_{i<r} |i, i\rangle$. The run is deterministic and takes under half a minute.

5 Scope and limitations

Theorem 1, Corollary 2 and Propositions 3 and 4 are proved analytically; the checker is a floating-point confirmation on finitely many cases. The result rules out only criteria that use quadrature moments of bounded degree, together with an energy bound, uniformly over all states. It says nothing against the certification of a Schmidt number for a particular state or family of states, such as Gaussian states, and it does not concern nonlinear data such as purity. The separable partner σ_r is mixed; we have not examined whether a pure product state, or a pure state of smaller Schmidt rank, can share the low-order moments of $|\psi_r\rangle$.

Proposition 4 shows that at bounded energy the obstruction needs a reduced-state eigenvalue λ_r of at most $E/(r(r-1))$. A better-posed question is therefore the following: given an energy bound and a lower bound on the r-th Schmidt coefficient, or on the weight of a state beyond Schmidt rank $r-1$, which order of quadrature moments suffices to certify Schmidt number at least r? We leave it open.

The construction is elementary, and we make no claim of priority for it. Our search was limited to the works cited here, in which we did not find it stated; observations of the same kind may well appear elsewhere in the literature on moment-based entanglement criteria.

References

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