Exact ensemble-steering defect under partial environmental access

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17 November 2026; revised 19 November 2026

Abstract

Quantum steering allows an observer to prepare different ensembles for a distant system by choosing a measurement on a system that purifies it. This control is complete when the observer holds the whole purifier, but it can fail when part of the purifying information has escaped into an inaccessible environment. We determine exactly how large this failure is for a qubit state with two branches. If the branches have weights p and q and their hidden environmental records have overlap κ, the worst-case ensemble-preparation error is $(1-\kappa)\sqrt{pq}$. Thus the lost steering power is precisely the branch coherence recorded outside the observer's reach. A binary target ensemble and a two-outcome measurement attain this minimax value; when the hidden records are orthogonal, the value reduces to half the known ℓ_1 coherence of assistance of a qubit. For a chain of summable controlled rotations, the same formula gives matching spatial bounds and an asymptotically sharp geometric decay rate. The state is also the unique ground state of a gapped quasi-local parent Hamiltonian. The proofs are short and use standard tools; the contribution is the exact value of this operational defect in an explicit model, and no general quantitative theory of partial steering is claimed.

1 Introduction

Quantum steering is the ability to control remotely how a mixed state is prepared. For example, suppose that Alice and Bob share the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$. If Bob measures his qubit in the computational basis, Alice receives either $|0\rangle$ or $|1\rangle$. If he measures in the diagonal basis, she instead receives either $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ or $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. Alice's average state is $I/2$ in both cases; Bob's choice changes its ensemble decomposition, not the average state itself.

This freedom extends beyond the Bell pair. A mixed state admits many ensemble decompositions, and the Schrödinger–HJW theorem (after Schrödinger, and Hughston, Jozsa and Wootters) shows that an observer holding its entire purifying system can prepare every finite decomposition into pure states by measuring the purifier in a suitable basis, after enlarging it if necessary [1]. Decompositions into mixed states are coarse-grainings of pure ones and are prepared by merging outcomes. Perfect steering therefore relies on access to all the information that purifies the state.

The question studied here is what remains possible when that access is only partial. We split the purifier into a subsystem B that Bob can measure and a hidden environment C. If C retains a record of the alternative branches, some ensembles of Alice's state can no longer be prepared from B. We measure the resulting loss by the largest, over all target ensembles, of the smallest trace-distance error achievable by a measurement on B.

For a qubit state with two branches, this worst-case error has a simple exact meaning. Let p and q be the branch weights and let κ be the overlap between their records in C. The steering defect is $(1-\kappa)\sqrt{pq}$: it vanishes when the hidden environment carries no branch information and grows with the coherence that has become inaccessible. The minimax formulation below makes this statement precise and identifies a binary target ensemble and a two-outcome measurement that attain the value. The same parameters acquire a spatial meaning when progressively larger portions of an environment become accessible.

The exact formula has two consequences. First, it relates the steering defect to a natural trace-distance defect for state-preserving conditional expectations. Second, it turns the tail of a summable controlled-rotation chain into explicit upper and lower bounds on the benefit of increasing environmental access. The resulting decay is quadratic in the small tail angles, although the unitary truncation error is linear.

For a state ρ_{AB} with marginal ρ_A, a target ensemble is a finite family of positive operators $\{\lambda_i\}$ satisfying $\sum_i \lambda_i = \rho_A$; the probabilities are included in the traces of the λ_i. We define

$$s_B(\rho) = \sup_{\{\lambda_i\}:\,\sum_i \lambda_i = \rho_A} \inf_{\{E_i\}\,\text{POVM on } B} \frac{1}{2} \sum_i \left\| \lambda_i - \mathrm{Tr}_B[(I \otimes E_i)\rho_{AB}] \right\|_1 . \tag{1}$$

The target and measurement outcomes carry the same labels, and the positive-operator-valued measure (POVM) may depend on the target ensemble. This quantity is distinct from robustness against local-hidden-state models: it asks how well one measurement system can realize every decomposition of a fixed marginal.

2 Related work

Schrödinger's steering observation was given an operational definition and placed in a strict hierarchy between entanglement and Bell nonlocality by Wiseman, Jones and Doherty [2]. Their question is whether the conditional states admit a local-hidden-state model. The minimax (1) asks a different, preparation-oriented question: how closely one fixed measurement system can realize every decomposition of a fixed marginal. Its starting point is the Schrödinger–HJW theorem, whose history and exact statement are reviewed by Kirkpatrick [1]; the defect measures how this theorem fails once part of the purifier is removed.

For two-qubit states, the steering ellipsoid is the set of Bloch vectors to which measurements on one qubit can steer the other [3]. Jevtic, Pusey, Jennings and Rudolph call steering complete when every convex decomposition of the marginal Bloch vector into points of this ellipsoid is realizable, and they show that this holds for every non-degenerate ellipsoid, hence for every entangled two-qubit state [3, Appendix C]. The state ρ_{AB} studied below is entangled for $\kappa > 0$ and is therefore completely steerable in their sense. Equation (1) also counts decompositions whose members lie outside the ellipsoid, and its exact value quantifies how far these targets are from every realizable ensemble.

The coherence of assistance supplies the extremal ensemble. Zhao, Ma, Quan, Fan and Pereira define the ℓ_1 coherence of assistance $C_a^{\ell_1}$ as the largest average ℓ_1 coherence over pure-state decompositions, prove $C_a^{\ell_1}(\rho) \le \sum_{i\neq j} \sqrt{\rho_{ii}\rho_{jj}}$ by the Cauchy–Schwarz inequality, and show that equality holds for qubits, with the binary ensemble $\sqrt{p}|0\rangle \pm \sqrt{q}|1\rangle$ optimal [4, Theorem 1, Corollary 1, Lemma 1]. When the hidden records are orthogonal ($\kappa = 0$), every state Bob prepares is diagonal, and Theorem 1 reduces to $s_B = \sqrt{pq} = C_a^{\ell_1}(\rho_A)/2$. The new ingredient for $\kappa > 0$ is the matching lower bound over all measurements on B.

Branch states whose decoherence factors are products of overlaps over subenvironments are standard in quantum Darwinism, where they serve to quantify the redundancy of records in

environment fragments [5, Secs. III–IV] and [6]. The spatial model of Section 5 uses the same structure for a different task: it measures the steering power gained as larger portions of the environment become accessible.

We did not find the minimax value of Theorem 1, the comparison with δ_* in Section 4, or the spatial rate of Corollary 2 in the literature consulted. Their proofs combine the standard tools recalled above.

3 Exact minimax result

Consider the pure state

$$|\Psi\rangle = \sqrt{p}|0\rangle_A|b_0\rangle_B|u\rangle_C + \sqrt{q}|1\rangle_A|b_1\rangle_B|v\rangle_C, \qquad p+q=1, \quad 0<q\le p, \tag{2}$$

where b_0, b_1 are orthonormal and u, v are unit vectors. The system C is inaccessible. A phase choice gives $\langle v|u\rangle = \kappa \in [0,1]$. On the logical support of B,

$$\rho_{AB} = p|00\rangle\langle 00| + q|11\rangle\langle 11| + \kappa\sqrt{pq}(|00\rangle\langle 11| + |11\rangle\langle 00|).$$

Compressing a physical POVM to this support gives a qubit POVM. Conversely, every logical POVM extends to the physical space by assigning the orthogonal complement to one outcome.

Theorem 1. *For the state in (2),*

$$\boxed{s_B(\rho) = (1-\kappa)\sqrt{pq}.}$$

The supremum is attained by $\lambda_\pm = |\phi_\pm\rangle\langle\phi_\pm|/2$, where $|\phi_\pm\rangle = \sqrt{p}|0\rangle \pm \sqrt{q}|1\rangle$.

Proof. Set $D = \operatorname{diag}(\sqrt{p}, \sqrt{q})$ and write $\lambda_i = \begin{pmatrix} x_i & z_i \\ \bar{z}_i & y_i \end{pmatrix}$. The logical POVM $E_i = (D^{-1}\lambda_i D^{-1})^T$ is positive and sums to I. It prepares

$$\mu_i = \begin{pmatrix} x_i & \kappa z_i \\ \kappa\bar{z}_i & y_i \end{pmatrix}.$$

Its cost is $(1-\kappa)\sum_i |z_i|$. Positivity and the Cauchy–Schwarz inequality give $\sum_i |z_i| \le \sum_i \sqrt{x_i y_i} \le \sqrt{pq}$, as in the qubit case of [4, Theorem 1], establishing an upper bound for every finite target ensemble.

For the matching lower bound, use the binary target stated in the theorem. Every effect $0 \le E \le I$ satisfies $|E_{01}| \le 1/2$, because $\|2E - I\| \le 1$. For $\Delta = \lambda_+ - \mu_+$,

$$\|\Delta\|_1 \ge \operatorname{Tr}(X\Delta) = \sqrt{pq} - 2\kappa\sqrt{pq}\,\Re E_{10} \ge (1-\kappa)\sqrt{pq}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The two differences are opposite because both ensembles sum to ρ_A; their total cost is therefore $\|\Delta\|_1$. The effects $E_\pm = (I \pm X)/2$ attain equality. □

4 Conditional-expectation defect

When Bob holds the whole purifier of ρ_{AC}, perfect steering of A from B would follow if ρ_{AC} were a product $\rho_A \otimes \tau$, that is, if the conditional expectation $E_\tau(X \otimes Y) = X \operatorname{Tr}(\tau Y) \otimes I$ of the matrix algebra of AC onto that of A preserved ρ_{AC}. Every conditional expectation of a full matrix algebra $M_A \otimes M_C$ onto $M_A \otimes I$ has this form for some state τ. Define the optimal state-preservation defect $\delta_* = \inf_\tau \frac{1}{2} \|\rho_{AC} - \rho_A \otimes \tau\|_1$. Here

$$\rho_{AC} = p|0\rangle\langle 0| \otimes |u\rangle\langle u| + q|1\rangle\langle 1| \otimes |v\rangle\langle v|.$$

Writing $U = |u\rangle\langle u|$ and $V = |v\rangle\langle v|$ gives

$$\begin{aligned} 2\delta_\tau &= p \|U - \tau\|_1 + q \|V - \tau\|_1 \\ &\geq q \|U - V\|_1 + (p - q) \|U - \tau\|_1 . \end{aligned}$$

The infimum is attained at $\tau = U$, and hence

$$\boxed{\delta_* = q\sqrt{1 - \kappa^2}, \qquad s_B = \sqrt{pq}\left(1 - \sqrt{1 - (\delta_*/q)^2}\right).} \tag{3}$$

Faithful states approximating U preserve the same infimum if faithfulness is required. The second equality assumes fixed q and $0 \leq \delta_* \leq q$.

At $\kappa = 0$ with $q \to 0$, one has $s_B = \sqrt{\delta_*(1 - \delta_*)}$, which rules out a uniform linear upper bound in δ_*. For fixed $q > 0$ and $\kappa \to 1$,

$$s_B \sim \frac{\sqrt{pq}}{2q^2}\delta_*^2.$$

The two limits identify different scaling regimes: a general square-root bound can be far from sharp when leakage is small and the Schmidt weights remain fixed.

5 Spatially increasing access

Place a qubit A, a flag qubit B_0, and qubits E_j for $j \geq 1$ in the state obtained from $(\sqrt{p}|00\rangle + \sqrt{q}|11\rangle)_{AB_0} \otimes_j |0\rangle_{E_j}$ by the controlled rotations

$$U_j = |0\rangle\langle 0|_A \otimes I + |1\rangle\langle 1|_A \otimes R(\theta_j), \qquad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Assume $0 \leq \theta_j < \pi/2$ and $\sum_j \theta_j < \infty$. The commuting product $U = \prod_j U_j$ converges in norm in the quasi-local algebra because $\|U_j - I\| = 2\sin(\theta_j/2) \leq \theta_j$. Bob controls $B_R = B_0 E_1 \cdots E_R$, and the remaining qubits form C_R.

Corollary 2. *For $V_R = \sum_{j>R} \theta_j^2$,*

$$s_{B_R} = \sqrt{pq}\left(1 - \prod_{j>R} \cos\theta_j\right), \tag{4}$$

$$\sqrt{pq}(1 - e^{-V_R/2}) \leq s_{B_R} \leq \frac{\sqrt{pq}}{2} V_R. \tag{5}$$

Consequently, $s_{B_R} \sim \sqrt{pq}V_R/2$ whenever $V_R \to 0$ through positive values. A vanishing tail gives zero defect exactly.

Proof. The inaccessible branches have overlap $\prod_{j>R}\cos\theta_j$, so Theorem 1 gives (4). For a finite sequence, $1-\prod_j a_j \le \sum_j (1-a_j)$ when $0 \le a_j \le 1$. The inequalities $1-\cos\theta \le \theta^2/2$ and $\cos\theta \le e^{-\theta^2/2}$ on $[0,\pi/2)$ then give (5); convergence permits passage to the infinite product. The ratio of the two bounds tends to one as $V_R \to 0$. □

For $\theta_j = \theta_0 r^j$ with $0 < r < 1$ and $0 < \theta_0 < \pi/2$,

$$s_{B_R} \sim \frac{\sqrt{pq}\,\theta_0^2}{2(1-r^2)} r^{2(R+1)}. \tag{6}$$

The unitary truncation obeys $\|U - U_{\le R}\| \le \theta_0 r^{R+1}/(1-r)$, so the ensemble task decays to second order while the operator-norm truncation error is first order. The sufficient access radius for $s_{B_R} \le \varepsilon$ is

$$R \ge \max\left\{0, \left\lceil \frac{\log[\sqrt{pq}\,\theta_0^2/(2(1-r^2)\varepsilon)]}{2|\log r|} - 1 \right\rceil\right\}.$$

The optimal binary measurement acts on the span of the two branches of B_R. It can be implemented by undoing the accessible controlled rotations and measuring B_0 in the eigenbasis of X.

6 A gapped quasi-local parent model

On the product representation of the environmental qubits, define

$$H_0 = (I - |\psi_p\rangle\langle\psi_p|)_{AB_0} + \sum_{j\geq 1} |1\rangle\langle 1|_{E_j}, \qquad |\psi_p\rangle = \sqrt{p}|00\rangle + \sqrt{q}|11\rangle.$$

This positive excitation-number operator has a unique ground state and gap one. The unitarily equivalent operator $H = UH_0U^*$ on $U\mathcal{D}(H_0)$ has the same spectrum and the state above as its ground state. Each term $U|1\rangle\langle 1|_{E_j}U^* - |1\rangle\langle 1|_{E_j}$ is supported on $\{A, E_j\}$ and has norm at most $2\theta_j$. The transformed AB_0 term is quasi-local, with truncation error at most $2\sum_{j>R}\theta_j$. Thus the parent model is well defined for a summable angle sequence.

The interaction is inhomogeneous and long ranged, although summable. Its ground state is related to a block-product state by a quasi-local unitary, and no topological order is established. The partition A, B_R, C_R keeps Alice's system fixed while Bob's region grows, so the decay rate describes the gain from enlarging the controlled region, not the dependence of steering on the distance between two fixed observers. The model therefore supplies an exactly solvable example, while a general quantitative theory of steering under partial access, for instance in lattice systems with topological order, remains open.

7 Verification and scope

The public Python checker verifies the partial-trace preparations, the binary witnesses, rational test cases, spatial tails, and spectra of finite parent Hamiltonians; its recorded output is included in the same public entry. These computations test finite instances and complement the analytic proofs, which alone establish the universal statements. Bibliographic priority is a separate question: the search reported in Section 2 found the ingredients of the proofs but not the exact minimax value, which remains open to independent assessment.

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