The variance of a sum of independent quantum errors: a proof for centred errors and counterexamples without centring

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Abstract

When a quantum computation suffers several independent errors, one would like to predict the size of the total error from the sizes of its parts. Lacalle and Pozo Coronado model an error as a random pure state scattered around the error-free state on the unit sphere of the state space, measure its size by the mean squared distance V to the error-free state, and prove that two independent isotropic errors combine as $V_{12} = V_1 + V_2 - \frac{1}{2}V_1V_2$. They conjecture the same formula for arbitrary independent errors and state it as a theorem on the circle. We show that the answer depends on one hypothesis that the conjecture leaves implicit: whether the errors are centred, in the sense that their Euclidean mean points along the error-free state. The formula is equivalent to the multiplicativity of the mean overlap $m = 1 - V/2$, and composing independent errors multiplies mean operators. Hence the formula holds for any number of independent centred errors, isotropic or not, in both natural definitions of the sum, and an exact identity gives the correction term in general. The multiplicativity itself is elementary and classical; the contribution of the note is to identify centring as the hypothesis on which the conjecture turns and to supply exact counterexamples. Without centring the formula fails: two independent errors on the circle with density $(1 + \sin\theta)/(2\pi)$, which belongs to the family used in the source's own proof, give $V_{12} = 5/2$ instead of 2, so the theorem on the circle is false as stated; two deterministic over-rotations $R_x(2\pi/3)$ of one qubit give $V_{12} = 3$ instead of $3/2$. The same mechanism proves the companion fidelity formula of Lacalle, Pozo Coronado and Martín-Cuevas when the second error has an isotropic second moment.

1 Introduction

Errors in a quantum computer accumulate. A gate that misses its target by a small amount leaves the register in a pure state close to, but not equal to, the intended one, and a long computation composes many such imperfections. A simple model treats each error as a random point on the unit sphere of state space, concentrated around the error-free state, and asks how the spread of the final state depends on the spread of the individual errors. Because the sphere is compact, spreads cannot simply add: a total error never exceeds the diameter of the sphere. Lacalle and Pozo Coronado [1] measured the spread by the mean squared distance to the error-free state, which they call the variance, and found that for two independent errors whose densities depend only on the distance to their centre (isotropic errors) the variances combine by the rule

$$V(X_1 + X_2) = V(X_1) + V(X_2) - \frac{V(X_1)\,V(X_2)}{2}. \tag{1}$$

They conjectured that the rule holds for any two independent errors, proved it for arbitrary independent random variables on the circle, and illustrated the general case by an exact computation on the seven-dimensional sphere of two qubits.

This note settles the conjecture by identifying the hypothesis on which it turns. The variance of an error depends only on the Euclidean mean of the random state, and composing independent errors multiplies mean operators. Rule (1) is the statement that the mean overlap with the error-free state is multiplicative, and this holds whenever the error-free state is an eigenvector of the mean operators, that is, whenever the errors are *centred*: their Euclidean mean lies along the error-free state. Isotropy guarantees centring but is far stronger than needed. The source paper itself assumes that the error-free state "represents the mean" of each error, and under that convention the conjecture is true for anisotropic errors with different distributions, any number of errors and every number of qubits, with a three-line proof (Theorems 2 and 3, Corollary 4). Without centring the rule is false, and the theorem on the circle fails already for a pair of densities from the family used in its proof (Proposition 5). The physically important case is that of biased coherent errors, such as a systematic over-rotation, for which the variance grows faster than the rule predicts.

The simplest illustration is a single qubit subject twice to the same deterministic over-rotation $R_x(2\pi/3) = \exp(-i\frac{\pi}{3}\sigma_x)$. Each rotation has overlap $\frac{1}{2}$ with $|0\rangle$, hence variance 1, and rule (1) predicts $\frac{3}{2}$ for the pair. The composite rotation $R_x(4\pi/3)$ has overlap $-\frac{1}{2}$, hence variance 3. The discrepancy comes from the transverse component of the mean, which is nonzero for a biased rotation and absent for a centred error; the general identity (3) below measures exactly this component.

2 Related work

The model and the conjecture are those of Lacalle and Pozo Coronado [1], published in 2019 and posted to arXiv in December 2024. They prove rule (1) for independent isotropic errors on every sphere S^d (their Theorem 2), through an explicit expansion of the densities in powers of $\cos\theta_0$, double-factorial integrals and a combinatorial identity on ordered trees. Their Theorem 3 asserts the rule for any two independent random variables on S^1. Its proof expands densities in the functions $g_k(\theta) \propto 1 + \cos^k\theta$ and $h_k(\theta) \propto 1 + \sin^k\theta$, treats the pairs (g_k, g_l), (g_k, h_l) and (h_k, g_l), and omits the pair (h_k, h_l); Proposition 5(a) shows that this pair violates the rule. Their Example on S^7 combines an anisotropic, non-centred first error with an isotropic second error and obtains $V = 445/224$; Section 4.3 explains why this example falls under Theorem 3 and therefore does not test the anisotropic second error.

Lacalle, Pozo Coronado and Martín-Cuevas [2] recall in 2026 that rule (1) "was proven for isotropic quantum errors and conjectured for general quantum errors" and pose a twin conjecture for the fidelity. Remark 6 proves that twin formula in the transport model when the second error has an isotropic second moment. We found no proof, refutation or correction of the variance conjecture in the literature examined (Section 6).

The distinction between centred and biased errors parallels the familiar contrast between stochastic and coherent noise in fault tolerance: errors of equal average size can have very different worst-case effects, and coherent errors appear to produce the largest gap between average-case and worst-case error [3]. Under composition, moreover, the infidelity of purely stochastic errors grows linearly in the number of gates, whereas that of purely unitary errors can grow quadratically, and the average gate fidelity of a composite channel is bounded between these two regimes [4]. Proposition 5(b) is an instance of this coherent accumulation in the variance language of [1].

The mechanism behind our positive results is classical in substance. On the circle, indepen-

dence of two angles gives $\mathbb{E}e^{i(\theta_1+\theta_2)} = \mathbb{E}e^{i\theta_1}\,\mathbb{E}e^{i\theta_2}$, the multiplicativity of the first trigonometric moment of a sum of independent angles, whose real part is the identity used in Proposition 5(a). In general, the mean operators of independent random isometries multiply, $\mathbb{E}[R_2R_1] = \mathbb{E}[R_2]\,\mathbb{E}[R_1]$, which is the defining-representation case of the fact that convolution of measures on a compact group multiplies their Fourier transforms. We claim no novelty for these facts; what the note adds is their application to the conjecture of [1], the identification of centring as the relevant hypothesis, and the exact correction term (3).

3 Setting

A pure state of n qubits, $\Psi = (x_0 + ix_1, \ldots, x_{d-1} + ix_d)$ in the computational basis, is a point of the real unit sphere $S^d \subset \mathbb{R}^{d+1}$ with $d = 2^{n+1} - 1$. We write $\langle\cdot,\cdot\rangle$ for the Euclidean inner product of $\mathbb{R}^{d+1}$; for two states, $\langle x, y\rangle = \mathrm{Re}\langle\Psi_x|\Psi_y\rangle$. The error-free state is $P = (1, 0, \ldots, 0)$, that is $|0\rangle$. An error is a random variable X on S^d, and its variance in the sense of [1] is

$$V(X) = \mathbb{E}\|X - P\|^2 = 2 - 2\,\mathbb{E}[x_0] \in [0, 4].$$

We define the *mean overlap* $m(X) = \mathbb{E}\langle P, X\rangle = \mathbb{E}[x_0] = 1 - V(X)/2$. Since $1 - \frac{1}{2}(V_1 + V_2 - \frac{1}{2}V_1V_2) = (1 - \frac{1}{2}V_1)(1 - \frac{1}{2}V_2)$, rule (1) is equivalent to

$$m(X_1 + X_2) = m(X_1)\, m(X_2). \tag{2}$$

Definition 1. An error X is *centred on* P if its Euclidean mean is parallel to P: $\mathbb{E}[X] = m(X)\,P$, that is, $\mathbb{E}[x_j] = 0$ for $1 \le j \le d$.

This is the precise content of the convention in [1] that P "represents the mean value" of the distribution. An isotropic error is centred: its law is invariant under the isometries of $\mathbb{R}^{d+1}$ that fix P (the rotations about the axis P when $d \ge 2$, the reflection $\theta \mapsto -\theta$ when $d = 1$), so $\mathbb{E}[X]$ is fixed by them and lies in $\mathbb{R}P$. The converse fails, since centring imposes d linear conditions on the law, whereas isotropy prescribes its whole angular dependence.

The source defines the sum $X_1 + X_2$ explicitly only when the second error is isotropic: the second error is then applied around the point reached by the first, with a density depending only on the distance to that point. Two definitions extend this to anisotropic errors.

Gate model. Each error is a random isometry $R_j \in O(d+1)$; for n qubits, a random unitary in $U(2^n) \subset O(2^{n+1})$, for instance a noisy gate composed with the inverse of the ideal gate. The isometries R_1, R_2 are independent, $X_j = R_jP$ is the error seen from the reference state, and $X_1 + X_2 := R_2R_1P$. This is the physical definition: the second error acts as an operator on the already erroneous state.

Transport model. The first error is a random point $X_1 \in S^d$; the second is a random point $Y_2 \in S^d$, independent of X_1, whose law is given around P. Choose a measurable section $T : S^d \to O(d+1)$ with $T(x)P = x$ and set $X_1 + X_2 := T(X_1)\,Y_2$: the law of the second error is copied around the point reached by the first. When Y_2 is isotropic the law of $T(x)Y_2$ does not depend on the choice of T, and this is the definition of [1].

4 Results

4.1 The identity behind the formula

Theorem 2 (gate model)**.** *Let R_1, R_2 be independent random isometries in $O(d+1)$, $X_j = R_jP$, $m_j = m(X_j)$ and $V_j = V(X_j)$. Let $u_1 = \mathbb{E}[X_1] - m_1P$ be the transverse mean of the*

first error and $w_2 = \mathbb{E}[R_2^{\mathsf{T}} P] - m_2 P$ *the transverse mean of the inverse error* $R_2^{-1} P$*. Then*

$$V(X_1 + X_2) = V_1 + V_2 - \frac{V_1 V_2}{2} - 2\,\langle w_2, u_1 \rangle. \tag{3}$$

Rule (1) *therefore holds if and only if* $\langle w_2, u_1 \rangle = 0$*; in particular it holds for every second error as soon as the first error is centred.*

Proof. We have $\langle P, R_2 R_1 P \rangle = \langle R_2^{\mathsf{T}} P, R_1 P \rangle$. The random vectors $R_2^{\mathsf{T}} P$ and $R_1 P$ are bounded and independent, and the inner product is bilinear, so $m(X_1 + X_2) = \langle \mathbb{E}[R_2^{\mathsf{T}} P], \mathbb{E}[R_1 P] \rangle$. Moreover $\langle P, \mathbb{E}[R_2^{\mathsf{T}} P] \rangle = \mathbb{E}\langle R_2 P, P \rangle = m_2$, so $\mathbb{E}[R_2^{\mathsf{T}} P] = m_2 P + w_2$ with $w_2 \perp P$, and likewise $\mathbb{E}[R_1 P] = m_1 P + u_1$ with $u_1 \perp P$. Hence $m(X_1 + X_2) = m_1 m_2 + \langle w_2, u_1 \rangle$, and $V = 2 - 2m$ gives (3). □

Theorem 3 (transport model)**.** *Let* X_1 *be any random point of* S^d*, let* Y_2 *be a random point of* S^d *independent of* X_1 *and centred on* P*, and let* T *be any measurable section. Then* $\mathbb{E}[T(X_1) Y_2 \mid X_1] = m(Y_2)\, X_1$*, and rule* (1) *holds for* $X_1 + X_2 = T(X_1) Y_2$*.*

Proof. Conditionally on X_1 the matrix $T(X_1)$ is fixed and Y_2 keeps its law, so $\mathbb{E}[T(X_1) Y_2 \mid X_1] = T(X_1)\mathbb{E}[Y_2] = m(Y_2)\, T(X_1) P = m(Y_2)\, X_1$. Taking expectations and the inner product with P gives $m(X_1 + X_2) = m(Y_2)\, m(X_1)$, which is (2). □

Corollary 4 (the conjecture under the centring convention)**.** *Let* $X_1, \dots, X_k$ *be mutually independent errors, each centred on* P*, isotropic or not, with equal or different laws. In either model,*

$$1 - \frac{V(X_1 + \cdots + X_k)}{2} = \prod_{j=1}^{k} \Big(1 - \frac{V(X_j)}{2}\Big).$$

In particular rule (1) *holds for* $k = 2$ *in every dimension* $d \geq 1$*, hence for every number of qubits.*

Proof. In the gate model, centring means $\mathbb{E}[R_j] P = m_j P$. By mutual independence $\mathbb{E}[R_k \cdots R_1] = \mathbb{E}[R_k] \cdots \mathbb{E}[R_1]$, and $\mathbb{E}[R_k] \cdots \mathbb{E}[R_1] P = m_k \cdots m_1 P$ by induction; take the inner product with P. In the transport model, let $X^{(j)}$ be the state after j errors. Theorem 3, applied conditionally on $X^{(j-1)}$, gives $\mathbb{E}[X^{(j)} \mid X^{(j-1)}] = m_j X^{(j-1)}$, hence $\mathbb{E}[X^{(k)}] = m_k \cdots m_1 P$ by the tower property. □

The main theorem of [1] is the special case of Corollary 4 in which every law is invariant under the isometries fixing P. Isotropy is used there only to make P an eigenvector of the mean operators, and centring is enough for that.

4.2 Counterexamples without centring

Proposition 5. *Rule* (1) *fails for independent errors that are not centred, in each of the following exact situations.*

(a) On the circle. *Let* X_1, X_2 *be independent with density* $h_1(\theta) = (1 + \sin\theta)/(2\pi)$ *on* S^1*. Then* $V(X_j) = 2$*, so rule* (1) *predicts* 2*, whereas* $V(X_1 + X_2) = 5/2$ *when the sum is the rotation* $\theta = \theta_1 + \theta_2$*, and* $3/2$ *with the convention* $\theta = \theta_2 - \theta_1$*. In either case Theorem 3 of [1] is false as stated.*

(b) One qubit, coherent errors. *For* $U_1 = U_2 = R_x(2\pi/3)$ *one has* $V_1 = V_2 = 1$ *and* $V(X_1 + X_2) = 3$*, whereas rule* (1) *predicts* $3/2$*.*

(c) Centring the second error is not enough in the gate model. *On one qubit, let U_2 equal $R_y(\pi/2)$ or $R_y(-\pi/2)Z$ with probability $\frac{1}{2}$ each, and let $X_1 = R_y(\pi/2)|0\rangle$ be deterministic. Then the second error is centred with $m_2 = \frac{\sqrt{2}}{2}$, $m_1 = \frac{\sqrt{2}}{2}$, and $m(X_1+X_2) = 0 \neq \frac{1}{2}$, so $V(X_1 + X_2) = 2$ instead of 1.*

Proof. (a) The density h_1 is nonnegative with total mass 1, and $\mathbb{E}\cos\theta = 0$, $\mathbb{E}\sin\theta = \frac{1}{2}$. Hence $m(X_j) = 0$ and $V(X_j) = 2$. By independence, $\mathbb{E}\cos(\theta_1 \pm \theta_2) = \mathbb{E}\cos\theta_1 \mathbb{E}\cos\theta_2 \mp \mathbb{E}\sin\theta_1 \mathbb{E}\sin\theta_2 = \mp\frac{1}{4}$, so $V(X_1+X_2) = 2 \pm \frac{1}{2}$. The density h_1 belongs to the family $\{h_k\}$ used in the proof of Theorem 3 of [1]; the case (h_k, h_l) is the one missing from that proof. On S^1 the correct statement is $V(X_1 + X_2) = V_1 + V_2 - \frac{1}{2}V_1V_2 + 2\,\mathbb{E}[\sin\theta_1]\,\mathbb{E}[\sin\theta_2]$ for the rotation convention, the special case of (3) with $u_1 = \mathbb{E}[\sin\theta_1]e_1$ and $w_2 = -\mathbb{E}[\sin\theta_2]e_1$.

(b) In the real picture, $\langle 0|R_x(\alpha)|0\rangle = \cos(\alpha/2)$, so $m_1 = m_2 = \cos(\pi/3) = \frac{1}{2}$ and $V_1 = V_2 = 1$. The product is $R_x(4\pi/3)$, with $m(X_1 + X_2) = \cos(2\pi/3) = -\frac{1}{2}$ and $V(X_1 + X_2) = 3$. More generally, for independent random over-rotations $R_x(\alpha_j)$ the identity (3) reads $V_{12} = V_1 + V_2 - \frac{1}{2}V_1V_2 + 2\,\mathbb{E}[\sin\frac{\alpha_1}{2}]\,\mathbb{E}[\sin\frac{\alpha_2}{2}]$: a systematic bias of the same sign makes the variance grow faster than rule (1), and the rule is exact when one of the angles has a symmetric law.

(c) With $R_y(\alpha) = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$ and $Z = \mathrm{diag}(1,-1)$, the mean operator is $M_2 = \mathbb{E}[U_2] = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, so $M_2|0\rangle = \frac{\sqrt{2}}{2}|0\rangle$ and the second error is centred, whereas $M_2^{\mathsf{T}}|0\rangle = \frac{\sqrt{2}}{2}(|0\rangle - |1\rangle)$ has a transverse part. With $X_1 = \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$, independence gives $m(X_1+X_2) = \langle 0|M_2X_1 = \frac{1}{2}(1-1) = 0$, while $m_1m_2 = \frac{1}{2}$. This agrees with (3): $u_1 = \frac{\sqrt{2}}{2}|1\rangle$, $w_2 = -\frac{\sqrt{2}}{2}|1\rangle$ and $\langle w_2, u_1\rangle = -\frac{1}{2}$. □

In the gate model it is therefore the centring of the first error, or of the inverse second error $R_2^{-1}P$, that matters. In the transport model, when the second error is not centred, the variance of the sum depends in addition on the choice of the section T, so that the sum is not determined by the laws of X_1 and X_2 alone; the checker of Section 5 exhibits finite ensembles for which two sections give variances differing by more than 1.

4.3 The example on the seven-sphere

The example illustrating the conjecture in [1] uses hyperspherical angles $(\theta_0, \theta_1, \theta_2, \dots)$ on S^7, with $x_0 = \cos\theta_0$ and $x_1 = \sin\theta_0\cos\theta_1$, and takes the densities

$$g_1 \propto 2 + \cos\theta_0 \sin^2\theta_1 + \cos\theta_1 \sin\theta_2, \qquad g_2 \propto 1 + \cos\theta_0$$

for the first and the second error. The density g_2 is isotropic, hence centred. The density g_1 is anisotropic and not centred: an exact integration gives $\mathbb{E}[x_1] = 32768/(55125\,\pi^2) \approx 0.0602$. The example therefore falls under Theorem 3, which gives $V = 2 - 2m_1m_2$ with $m_1 = \frac{3}{56}$ and $m_2 = \frac{1}{8}$, that is $V = \frac{445}{224} \approx 1.98661$, the value obtained in [1] by a sixfold integral. It tests neither an anisotropic nor a non-centred second error.

4.4 The companion fidelity formula

Remark 6. Lacalle, Pozo Coronado and Martín-Cuevas [2] conjecture, for $F^2(X) = \mathbb{E}[x_0^2 + x_1^2] = \mathbb{E}|\langle 0|\Psi_X\rangle|^2$ and $f_0^2 = 2/(d+1)$, that

$$F^2(X_1 + X_2) = f_0^2 + \frac{(F^2(X_1) - f_0^2)(F^2(X_2) - f_0^2)}{1 - f_0^2}.$$

In the transport model this holds, for an arbitrary first error and any section, whenever the second error has an isotropic second moment, $\mathbb{E}[Y_2 Y_2^{\mathsf{T}}] = a\,PP^{\mathsf{T}} + b\,(\mathbf{1} - PP^{\mathsf{T}})$ with $a + db = 1$.

Proof. Conditionally on X_1, $\mathbb{E}[X X^{\mathsf{T}} \mid X_1] = T(X_1)\mathbb{E}[Y_2 Y_2^{\mathsf{T}}]T(X_1)^{\mathsf{T}} = a\,X_1 X_1^{\mathsf{T}} + b\,(\mathbf{1} - X_1 X_1^{\mathsf{T}})$, independently of the section. Taking the sum of the $(0,0)$ and $(1,1)$ entries and then expectations, $F^2(X_1 + X_2) = (a-b)F^2(X_1) + 2b$. Moreover $F^2(X_2) = a + b$, and $a + db = 1$ implies $1 - (a-b) = (d+1)b$, so that $2b = f_0^2(1-(a-b))$ and $a + b - f_0^2 = (a-b)(1-f_0^2)$. Substituting gives the formula. □

The checker shows that the fidelity formula fails for a generic second error in the transport model. The gate model is not treated here.

5 Verification

The statements above are proved analytically. The public Python checker tests them, and its recorded output is part of the same entry. In exact arithmetic (SymPy) it recomputes the three counterexamples of Proposition 5 and the example of Section 4.3: $V(g_1) = \frac{53}{28}$, $V(g_2) = \frac{7}{4}$, $2 - 2m_1 m_2 = \frac{445}{224}$ and $\mathbb{E}[x_1] \neq 0$ under g_1. A Monte Carlo simulation of that example in the transport model, with about 3×10^6 samples and a fixed seed, agrees with $\frac{445}{224}$ within four standard errors and excludes the value 2. In floating-point arithmetic the checker verifies the identity (3) to 10^{-12} on 360 random finite ensembles of anisotropic, non-identically distributed isometries in $O(2)$, $O(4)$ and $O(8)$, the exactness of rule (1) when the first error is centred and its generic failure (deviations up to 0.67), Theorem 3 on 45 finite ensembles on S^1, S^3 and S^7 with two different sections, and Remark 6 on S^3 and S^7. The run takes a few seconds.

6 Scope and limitations

Analytic proof. Theorems 2 and 3, Corollary 4, Proposition 5 and Remark 6 are proved in full; the remark only in the transport model.

What this says about the conjecture. The conjecture specifies neither the sum of anisotropic errors nor whether centring is assumed. Read with the centring convention stated in [1], it is true in both natural models, and the proof uses neither isotropy nor equality of the laws. Read without that convention, as the theorem on the circle and the example on S^7 suggest, since both use non-centred densities, it is false, and the theorem on the circle must be restricted to pairs in which one error is centred. The weakest sufficient condition in the gate model is $\langle w_2, u_1 \rangle = 0$. Independence enters only through the factorisation $\mathbb{E}[R_2^{\mathsf{T}} P \otimes R_1 P] = \mathbb{E}[R_2^{\mathsf{T}} P] \otimes \mathbb{E}[R_1 P]$; for k errors the expectation of the full product must factorise, which pairwise independence does not ensure.

Physical scope. Centring excludes biased coherent errors, such as a systematic over-rotation, for which the variance grows faster than rule (1) predicts, as in Proposition 5(b). The threshold estimate that the source derives from the rule, a variance per gate below C/k for k gates, relies on the absence of such bias. The variance V is sensitive to the global phase; the "quantum variance" $V_q = 2 - 2\,\mathbb{E}\sqrt{x_0^2 + x_1^2}$ introduced in [1] to remove this dependence is not a function of $\mathbb{E}[X]$, and none of the results above concerns it. The two models of the sum cover the natural readings of the source without exhausting every conceivable definition, for instance a Markov kernel whose shape depends on the starting point.

Bibliographic priority. The source was read in its arXiv TEX file, and the 2026 companion paper in its published version. The works citing the source were listed through the Semantic

Scholar interface and complemented by web searches and an OpenAlex query; none of them proves, refutes or corrects the conjecture, and its authors still describe it as conjectural in January 2026. Priority is claimed only for the application to this conjecture, the role of centring, the identity (3) and the counterexamples, within the limits of this documented search. The underlying multiplicativity of trigonometric moments and of mean operators is classical (Section 2), and the isotropic case of the source is a special case of that multiplicativity.

References

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